GEOMETRY REVISITED

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19

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GEOMETRY REVISITED

To

Our Grandchildren

Preface

He who despises Euclidean Geometry is like a man who, returning from foreign parts, disparages his home.

H. G. Forder.

The mathematics curriculum in the secondary school normally includes a single one-year course in plane geometry or, perhaps, a course in geometry and elementary analytic geometry called tenth-year mathematics. This course, presented early in the student's secondary school career, is usually his sole exposure to the subject. In contrast, the mathematically minded student has the opportunity of studying elementary algebra, intermediate algebra, and even advanced algebra. It is natural, therefore, to expect a bias in favor of algebra and against geometry. Moreover, misguided enthusiasts lead the student to believe that geometry is "outside the main stream of mathematics" and that analysis or set theory should supersede it.

Perhaps the inferior status of geometry in the school curriculum stems from a lack of familiarity on the part of educators with the nature of geometry and with advances that have taken place in its development. These advances include many beautiful results such as Brianchon's Theorem (Section 3.9), Feuerbach's Theorem (Section 5.6), the Petersen–Schoute Theorem (Section 4.8) and Morley's Theorem (Section 2.9). Historically, it must be remembered that Euclid wrote for mature persons preparing for the study of philosophy. Until our own century, one of the chief reasons for teaching geometry was that its axiomatic method was considered the best introduction to deductive reasoning. Naturally, the formal method was stressed for effective educational purposes. However, neither ancient nor modern geometers have hesitated to adopt less orthodox methods when it suited them. If trigonometry, analytic geometry, or vector methods will help, the geometer will use them. Moreover, he has invented modern techniques of his own that

are elegant and powerful. One such technique is the use of *transformations* such as rotations, reflections, and dilatations, which provide shortcuts in proving certain theorems and also relate geometry to crystallography and art. This "dynamic" aspect of geometry is the subject of Chapter 4. Another "modern" technique is the method of *inversive* geometry, which deals with points and circles, treating a straight line as a circle that happens to pass through "the point at infinity". Some flavor of this will be found in Chapter 5. A third technique is the method of *projective* geometry, which disregards all considerations of distance and angle but stresses the analogy between points and lines (whole infinite lines, not mere segments). Here not only are any two points joined by a line, but any two lines meet at a point; parallel lines are treated as lines whose common point happens to lie on "the line at infinity". There will be some hint of the content of this subject in Chapter 6.

Geometry still possesses all those virtues that the educators ascribed to it a generation ago. There is still geometry in nature, waiting to be recognized and appreciated. Geometry (especially projective geometry) is still an excellent means of introducing the student to axiomatics. It still possesses the esthetic appeal it always had, and the beauty of its results has not diminished. Moreover, it is even more useful and necessary to the scientist and practical mathematician than it has ever been. Consider, for instance, the shapes of the orbits of artificial satellites, and the four-dimensional geometry of the space-time continuum.

Through the centuries, geometry has been growing. New concepts and new methods of procedure have been developed: concepts that the student will find challenging and surprising. Using whatever means will best suit our purposes, let us revisit Euclid. Let us discover for ourselves a few of the newer results. Perhaps we may be able to recapture some of the wonder and awe that our first contact with geometry aroused.

The authors are particularly grateful to Dr. Anneli Lax for her patient cooperation and many helpful suggestions.

H. S. M. C.

S. L. G.

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CHAPTER 1

Points and Lines Connected with a Triangle

> With a literature much vaster than those of algebra and arithmetic combined, and at least as extensive as that of analysis, geometry is a richer treasure house of more interesting and half-forgotten things, which a hurried generation has no leisure to enjoy, than any other division of mathematics.
>
> *E. T. Bell*

The purpose of this chapter is to recall some of these half-forgotten things to which Dr. Bell referred, to derive some new theorems, developed since Euclid, and to apply our findings to interesting situations. We consider an arbitrary triangle and its most famous associated points and lines: the circumcenter, medians, centroid, angle-bisectors, incenter, excenters, altitudes, orthocenter, Euler line, and nine-point center.

The angle-bisectors lead naturally to a digression on the Steiner–Lehmus theorem, which was believed for a hundred years to be difficult to prove, though we see now that it is really quite easy.

Finally, from a triangle and a point P of general position, we derive a new triangle whose vertices are the feet of the perpendiculars from P to the sides of the given triangle. This idea leads to some amusing developments, some of which are postponed till the next chapter.

1.1 The extended Law of Sines

The Law of Sines is one trigonometric theorem that will be used frequently. Unfortunately, it usually appears in texts in a truncated form that is not so useful as an extended theorem could be. We take the liberty, therefore, of proving the Law of Sines in the form that we desire.

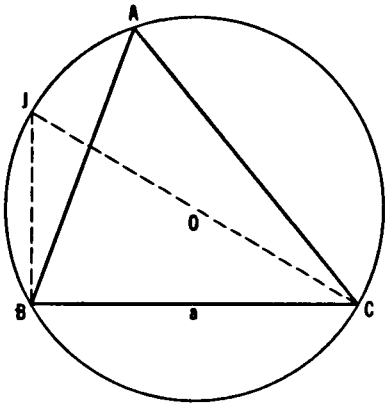


Figure 1.1A

We start with $\triangle ABC$ (labeled in the customary manner) and circumscribe about it a circle with center at O and with radius equal to R units, as shown in Figures 1.1A and 1.1B. We draw the diameter CJ, and the chord BJ.† In both of the situations shown, $\angle CBJ$ is a right angle, since it is inscribed in a semicircle. Hence, in both figures,

$$\sin J = \frac{a}{CJ} = \frac{a}{2R}.$$

In Figure 1.1A, $\angle J = \angle A$, because both are inscribed in the same arc of the circle. In Figure 1.1B, $\angle J = 180° - \angle A$, because opposite angles of an inscribed quadrilateral are supplementary. Remembering that $\sin \theta = \sin (180° - \theta)$, it follows that $\sin J = \sin A$ in both figures. Therefore, in either case, $\sin A = a/2R$, that is,

$$\frac{a}{\sin A} = 2R.$$

The same procedure, applied to the other angles of $\triangle ABC$, yields

$$\frac{b}{\sin B} = 2R, \qquad \frac{c}{\sin C} = 2R.$$

Combining results, we may state the extended Law of Sines thus:

THEOREM 1.11. *For a triangle ABC with circumradius R,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

† For typographic reasons, the length of a line segment with endpoints X and Y will be denoted simply by XY in this book.

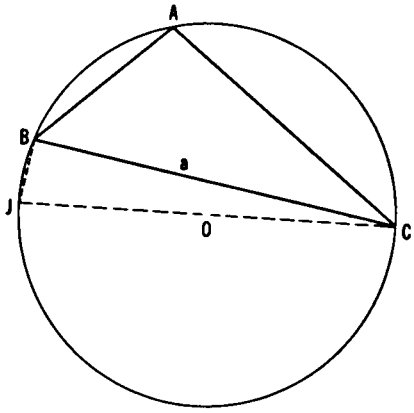


Figure 1.1B

Let us agree to denote the *area* of any figure by the name of the figure enclosed in parentheses. Thus (ABC) denotes the area of $\triangle ABC$, $(PQRS)$ denotes the area of a quadrilateral $PQRS$, and so on.

EXERCISES

1. Show that,† for any triangle ABC, even if B or C is an obtuse angle, $a = b \cos C + c \cos B$. Use the Law of Sines to deduce the "addition formula"
$$\sin (B + C) = \sin B \cos C + \sin C \cos B.$$

2. In any triangle ABC,
$$a (\sin B - \sin C) + b (\sin C - \sin A) + c (\sin A - \sin B) = 0.$$

3. In any triangle ABC, $(ABC) = abc/4R$.

4. Let p and q be the radii of two circles through A, touching BC at B and C, respectively. Then $pq = R^2$.

† In subsequent exercises we shall save space by omitting the words "Show that" or "Prove that". Thus any exercise appearing in the form of a theorem is intended to be *proved*.

1.2 Ceva's theorem

The line segment joining a vertex of a triangle to any given point on the opposite side is called a *cevian*. Thus, if X, Y, Z are points on the respective sides BC, CA, AB of triangle ABC, the segments AX, BY, CZ are cevians. This term comes from the name of the Italian mathematician Giovanni Ceva, who published in 1678 the following very useful theorem:

THEOREM 1.21. *If three cevians AX, BY, CZ, one through each vertex of a triangle ABC, are concurrent, then*

$$\frac{BX}{XC}\,\frac{CY}{YA}\,\frac{AZ}{ZB} = 1.$$

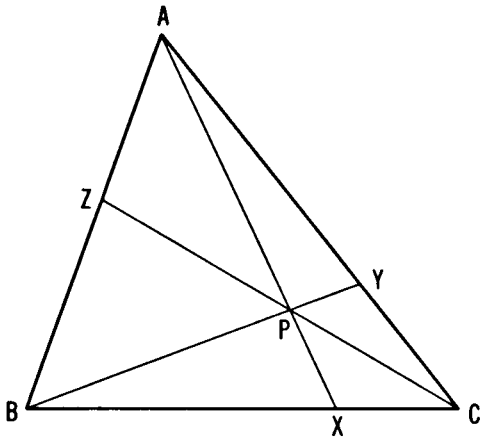


Figure 1.2A

When we say that three lines (or segments) are *concurrent*, we mean that they all pass through one point, say P. To prove Ceva's theorem, we recall that the areas of triangles with equal altitudes are proportional to the bases of the triangles. Referring to Figure 1.2A, we have

$$\frac{BX}{XC} = \frac{(ABX)}{(AXC)} = \frac{(PBX)}{(PXC)} = \frac{(ABX) - (PBX)}{(AXC) - (PXC)} = \frac{(ABP)}{(CAP)}.$$

Similarly,

$$\frac{CY}{YA} = \frac{(BCP)}{(ABP)}, \qquad \frac{AZ}{ZB} = \frac{(CAP)}{(BCP)}.$$

Now, if we multiply these, we find

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = \frac{(ABP)}{(CAP)}\frac{(BCP)}{(ABP)}\frac{(CAP)}{(BCP)} = 1.$$

The converse of this theorem holds also:

THEOREM 1.22. *If three cevians* AX, BY, CZ *satisfy*

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1,$$

they are concurrent.

To see this, suppose that the first two cevians meet at P, as before, and that the third cevian through this point P is CZ'. Then, by Theorem 1.21,

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ'}{Z'B} = 1.$$

But we are assuming

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1.$$

Hence

$$\frac{AZ'}{Z'B} = \frac{AZ}{ZB},$$

Z' coincides with Z, and we have proved that AX, BY, CZ are concurrent [9, p. 54].

EXERCISES

1. If X, Y, Z are the midpoints of the sides, the three cevians are concurrent.

2. Cevians perpendicular to the opposite sides are concurrent.

3 Let ABC and $A'B'C'$ be two non-congruent triangles whose sides are respectively parallel, as in Figure 1.2B. Then the three lines AA', BB', CC' (extended) are concurrent. (Such triangles are said to be *homothetic*. We shall consider them further in Section 4.7.)

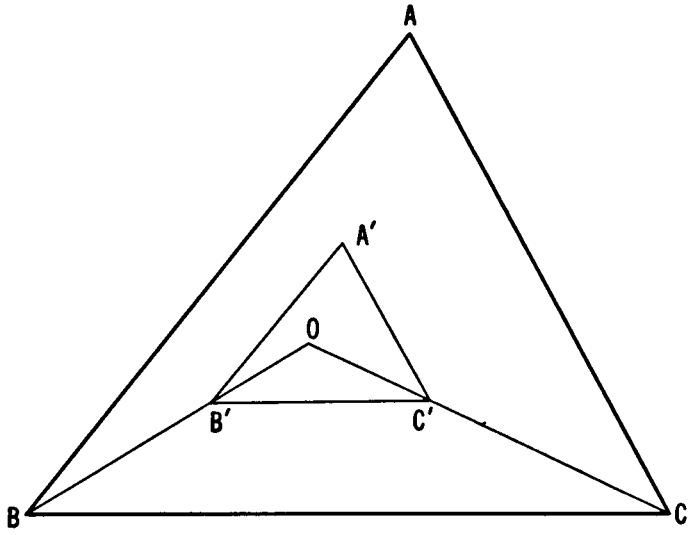


Figure 1.2B

4. Let AX be a cevian of length p, dividing BC into segments $BX = m$ and $XC = n$, as in Figure 1.2C. Then

$$a(p^2 + mn) = b^2m + c^2n.$$

Hint: Add expressions for the cosines of the two supplementary angles at X in terms of the sides of $\triangle ABX$ and $\triangle CAX$, respectively. This result is called *Stewart's theorem*, after M. Stewart, who stated it in 1746. It was probably discovered by Archimedes about 300 B.C., but the first known proof is by R. Simson, 1751.

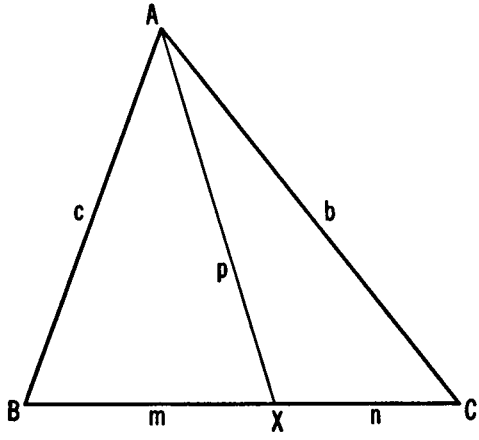


Figure 1.2C

1.3 Points of interest

There are many special points and lines connected with a triangle, and we shall have to restrict our attention to only a few of these. We have already referred to one such point, the center of the circle circumscribed about a triangle. We agree to call this point the *circumcenter* of the triangle, and we call the circle the *circumcircle* of the triangle. The circumcenter O is the intersection of the three perpendicular bisectors of the sides of the triangle (see Figure 1.3A). The radius of the circumcircle has already been denoted by the letter R.

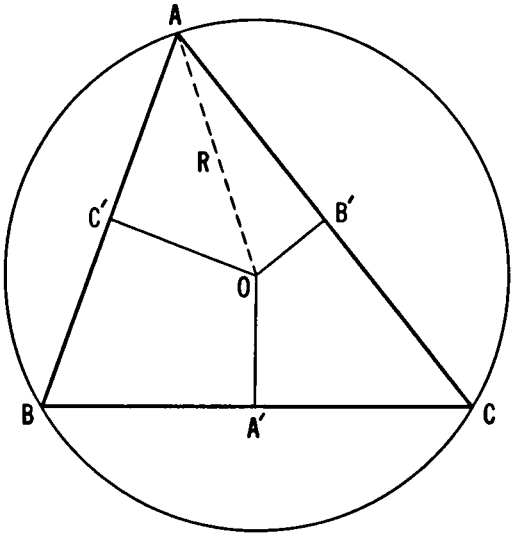


Figure 1.3A

The cevians that join the vertices of a triangle to the midpoints of the opposite sides are called *medians*. In Figure 1.3B, the lines AA', BB' and CC' are medians, so that $BA' = A'C$, $CB' = B'A$, and $AC' = C'B$. Applying Theorem 1.21, we conclude that the medians are concurrent. Their common point, G, is called the *centroid* of the triangle. Were a triangle to be cut out of material of uniform density, it would balance if suspended at this point, common to the medians. In other words, the centroid is the "center of gravity" of the triangle.

Looking again at Figure 1.3B, we are struck with the fact that $(GBA') = (GA'C)$, because the triangles have equal bases and the same altitude. That is why we have given the areas the same label, x. For the same reason, we have

$$(GCB') = (GB'A) \quad \text{and} \quad (GAC') = (GC'B),$$

so we have labeled these areas y and z as shown. However, we also have $(CAC') = (CC'B)$, that is, $2y + z = z + 2x$, whence $x = y$. Similarly, $(ABA') = (AA'C)$, whence $y = z$. Thus, we have shown that $x = y = z$, that is:

THEOREM 1.31. *A triangle is dissected by its medians into six smaller triangles of equal area.*

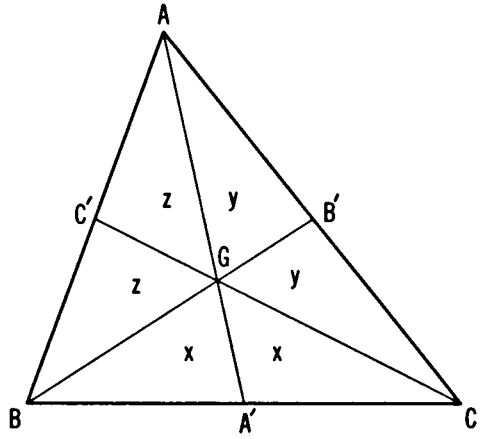


Figure 1.3B

Continuing our examination of Figure 1.3B, we again note that $(GAB) = 2(GBA')$. Since these triangles have the same altitude, it follows that $AG = 2GA'$. Similarly, $BG = 2GB'$, and $CG = 2GC'$:

THEOREM 1.32. *The medians of a triangle divide one another in the ratio* 2:1; *in other words, the medians of a triangle "trisect" one another.*

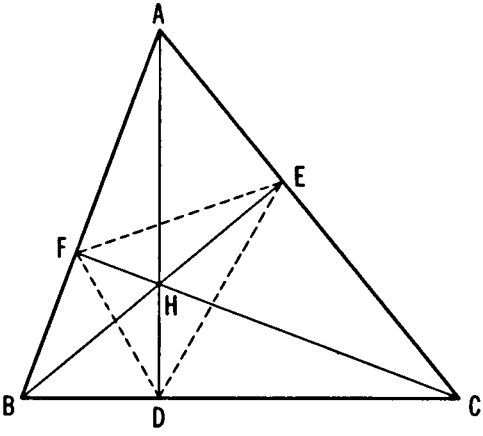


Figure 1.3C

The cevians AD, BE, CF (Figure 1.3C), perpendicular to BC, CA, AB, respectively, are called the *altitudes* of $\triangle ABC$. As we saw in the Exercise 2 of Section 1.2, the converse of Ceva's theorem shows them to be concurrent. Their common point H is called the *orthocenter*.†

The points D, E, F themselves are naturally called the *feet* of the altitudes. Joining them in pairs we obtain $\triangle DEF$, the *orthic triangle* of $\triangle ABC$.

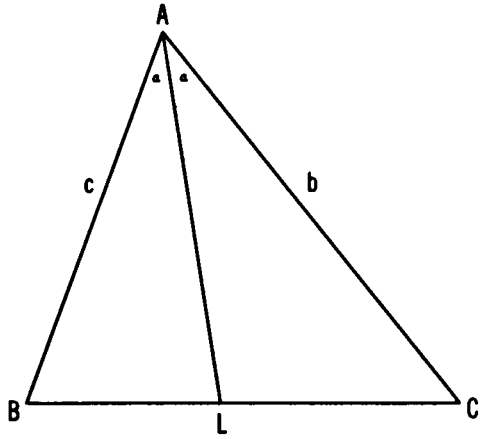


Figure 1.3D

Another important set of cevians are the three internal angle bisectors. Figure 1.3D shows one such bisector AL. Applying Theorem 1.11 to the two triangles ABL and ALC (whose angles at L, being supplementary, have equal sines), we obtain

$$\frac{BL}{\sin \frac{1}{2}A} = \frac{c}{\sin L}, \qquad \frac{LC}{\sin \frac{1}{2}A} = \frac{b}{\sin L},$$

whence

$$\frac{BL}{LC} = \frac{c}{b}.$$

Since we can derive similar results involving the internal bisectors of the angles B and C, we have now proved:

THEOREM 1.33. *Each angle bisector of a triangle divides the opposite side into segments proportional in length to the adjacent sides.*

† For the history of this term, see J. Satterly, Mathematical Gazette **45** (1962), p. 51.

Any point on AL (Figure 1.3D) is equidistant from CA and AB. Similarly, any point on the internal bisector of the angle B is equidistant from AB and BC. Hence the point I where these two bisectors meet is at equal distances r from all three sides:

THEOREM 1.34. *The internal bisectors of the three angles of a triangle are concurrent.*

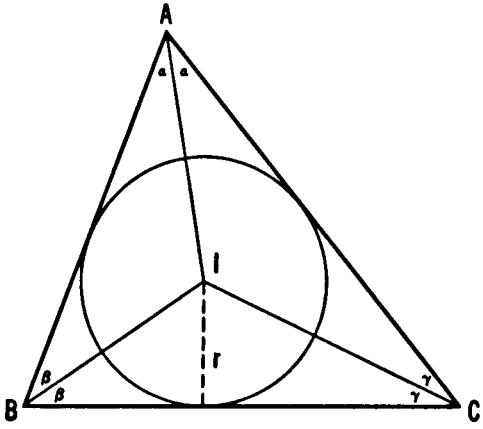


Figure 1.3E

The circle with center I and radius r (Figure 1.3E) has all three sides for tangents and is thus the inscribed circle or *incircle*. We call I the *incenter* and r the *inradius*.

EXERCISES

1. The circumcenter and orthocenter of an obtuse-angled triangle lie outside the triangle.

2. Find the ratio of the area of a given triangle to that of a triangle whose sides have the same lengths as the medians of the original triangle.

3. Any triangle having two equal medians is isosceles.

4. Any triangle having two equal altitudes is isosceles.

5. Use Theorems 1.22 and 1.33 to obtain another proof of Theorem 1.34.

6. Find the length of the median AA' (Figure 1.3B) in terms of a, b, c. *Hint*: Use Stewart's theorem (Exercise 4 of Section 1.2).

7. The square of the length of the angle bisector AL (Figure 1.3D) is

$$bc\left[1 - \left(\frac{a}{b+c}\right)^2\right].$$

8. Find the length of the internal bisector of the right angle in a triangle with sides 3, 4, 5.

9. The product of two sides of a triangle is equal to the product of the circumdiameter and the altitude on the third side.

1.4 The incircle and excircles

Figure 1.4A shows the incircle touching the sides BC, CA, AB at X, Y, Z. Since two tangents to a circle from any external point are equal, we see that $AY = AZ$, $BZ = BX$, $CX = CY$. We have accordingly labeled these segments x, y, z, so that

$$y + z = a, \quad z + x = b, \quad x + y = c.$$

Adding these equations and using Euler's labor-saving abbreviation s for the semiperimeter, we have

$$2x + 2y + 2z = a + b + c = 2s,$$

so that

$$x + y + z = s$$

and

THEOREM 1.41. $x = s - a, \quad y = s - b, \quad z = s - c.$

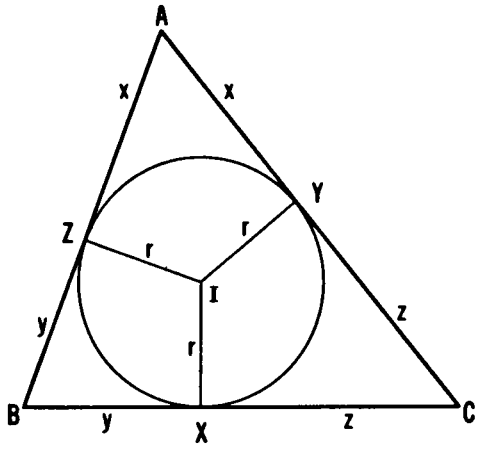


Figure 1.4A

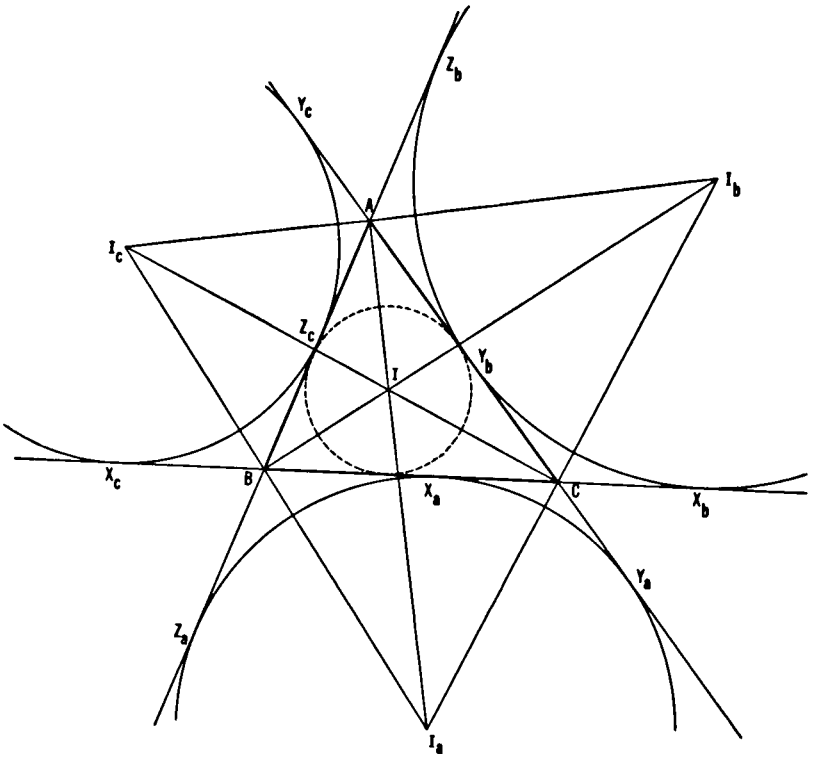


Figure 1.4B

Since the triangle IBC has base a and altitude r, its area is $(IBC) = \frac{1}{2}ar$. Adding to this the analogous expressions for (ICA) and (IAB), we obtain $\frac{1}{2}(a + b + c)r = sr$. Hence

THEOREM 1.42. $(ABC) = sr$.

Figure 1.4B shows the triangle $I_aI_bI_c$ whose sides are the external bisectors of the angles A, B, C. Any point on the bisector I_cI_a of $\angle B$ is equidistant from AB and BC. Similarly, any point on I_aI_b is equidistant from BC and CA. Hence the point I_a where these two external bisectors meet is at equal distances r_a from all three sides. Since I_a is equidistant from sides AB and AC, it must lie on the locus of points equidistant from these lines; that is, it must lie on the line AI, the internal bisector of $\angle A$:

THEOREM 1.43. *The external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle.*

The circle with center I_a and radius r_a, having all three sides for tangents, is one of the three "escribed" circles or *excircles*. We call their centers I_a, I_b, I_c the *excenters* and their radii r_a, r_b, r_c the *exradii*. Each excircle touches one side of the triangle internally and the other two sides (extended) externally. The incircle and the three excircles, each touching all three sides, are sometimes called the four *tritangent* circles of the triangle.

Marking the points of contact as in Figure 1.4B, we observe that, since two tangents from a point to a circle are equal in length,

$$BX_b = BZ_b$$

and

$$\begin{aligned} BX_b + BZ_b &= BC + CX_b + Z_bA + AB \\ &= BC + CY_b + Y_bA + AB = a + b + c = 2s. \end{aligned}$$

Thus the tangents from B (or any other vertex) to the excircle beyond the opposite side are of length s. Indeed,

$$AY_a = AZ_a = BZ_b = BX_b = CX_c = CY_c = s.$$

Also, since $CX_b = BX_b - BC = s - a$, and so on,

$$BX_c = BZ_c = CX_b = CY_b = s - a,$$

$$CY_a = CX_a = AY_c = AZ_c = s - b,$$

$$AZ_b = AY_b = BZ_a = BX_a = s - c.$$

EXERCISES

1. If three circles with centers A, B, C all touch one another externally their radii are $s - a$, $s - b$, $s - c$.

2. If s, r, R have their usual meaning, $abc = 4srR$.

3. The cevians AX, BY, CZ (Figure 1.4A) are concurrent. (Their common point is called the *Gergonne* point of $\triangle ABC$.)

4. $\triangle ABC$ is the orthic triangle of $\triangle I_aI_bI_c$ (Figure 1.4B.)

5. $(ABC) = (s - a)r_a = (s - b)r_b = (s - c)r_c$. (Cf. Theorem 1.42.)

6. $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$.

1.5 The Steiner-Lehmus theorem

There are a number of geometric problems that seem to exert a peculiar fascination on anybody who happens to stumble on them. This appears to have been a characteristic of geometry even in ancient times. One has only to recall the three famous problems of antiquity—the duplication of the cube, the trisection of the general angle, and the squaring of the circle. Attempts to solve these problems led to the development of many new branches of mathematics. Even now, there are would-be-mathematicians who send in "solutions" for these problems and dare the reader to prove them wrong.

One theorem that always excites interest may be stated thus:

THEOREM 1.51. *Any triangle that has two equal angle bisectors (each measured from a vertex to the opposite side) is isosceles.*

In 1840, this theorem was sent in a letter from C. L. Lehmus to C. Sturm, with a request for a pure geometric proof. Sturm mentioned it to a number of mathematicians. One of the first to answer the challenge was the great Swiss geometer Jacob Steiner, and it became known as the Steiner–Lehmus theorem. Papers on it appeared in various journals in 1842, 1844, 1848, almost every year from 1854 till 1864, and with a good deal of regularity during the next hundred years.

One of the simplest proofs makes use of the following two lemmas.

LEMMA 1.511. *If two chords of a circle subtend different acute angles at points on the circle, the smaller angle belongs to the shorter chord.*

PROOF. Two equal chords subtend equal angles at the center and equal angles (half as big) at suitable points on the circumference. Of two unequal chords, the shorter, being farther from the center, subtends a smaller angle there and consequently a smaller acute angle at the circumference.

LEMMA 1.512. *If a triangle has two different angles, the smaller angle has the longer internal bisector.* [5, p. 72.]

PROOF. Let ABC be the triangle, with $B < C$ as in Figure 1.5A;† let BM and CN bisect the angles B and C. We wish to prove that $BM > CN$. Take M' on BM so that $\angle M'CN = \frac{1}{2}B$. Since this is equal to $\angle M'BN$, the four points N, B, C, M' lie on a circle.

† Here and in what follows, we often denote the angle at B simply by the letter B.

Since

$$B < \tfrac{1}{2}(B + C) < \tfrac{1}{2}(A + B + C),$$

$$\angle CBN < \angle M'CB < 90°.$$

By Lemma 1.511, $CN < M'B$. Hence $BM > BM' > CN$.

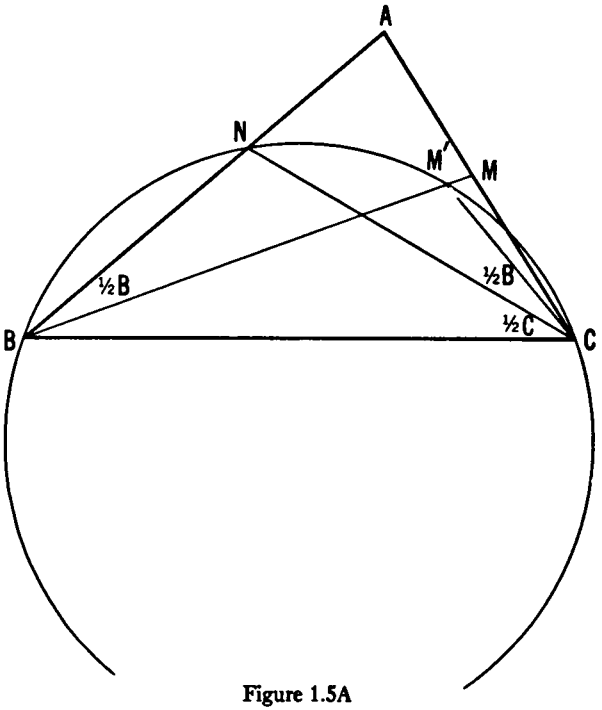


Figure 1.5A

PROOF OF THE THEOREM. It often happens that a theorem can be expressed in an equivalent "contrapositive" form. For instance, instead of saying *all men are mortal* we can just as well say *immortals are not men*. Instead of proving Theorem 1.51 itself, it will suffice for us to prove that *if, in* $\triangle ABC$, $B \neq C$, *then* $BM \neq CN$. But this is an immediate consequence of Lemma 1.512.

Archibald Henderson wrote one of the many biographies of Bernard Shaw, and also a tract on *The twenty-seven lines upon the cubic surface*. In his paper, *The Lehmus–Steiner–Terquem problem in global survey* (Scripta Mathematica, **21**, 1955, pp. 223–232, 309–312) he attributes a proof resembling ours to Lehmus himself (1850). The idea of replacing the theorem by a strengthened contrapositive appears in a paper by Victor Thébault (Mathesis, **44**, 1930, p. 97), who proved Lemma 1.512 exactly as above and then deduced Theorem 1.51 as a "corollary".

Henderson seems to have been slightly unhappy about Lehmus's proof, and about the earlier proof by Steiner, because they are not "direct". He would prefer to assume that $BM = CN$ without considering the situation when $B \neq C$. Most of the published proofs [e.g. **5**, p. 73] are likewise indirect. Several allegedly direct proofs [e.g. **6**, Answers to the Exercises, p. 2] have been proposed; but each of them is really an indirect proof in disguise. To see that this is the case, recall that only the very most elementary theorems are in practice proved completely. All the rest are proved with the aid of other theorems, already known: a whole chain of theorems going right back to the axioms. A proof cannot properly claim to be direct if any one of these auxiliary theorems has an indirect proof. Now, some of the simplest and most basic theorems have indirect proofs: consequently, if we insisted on complete directness, our store of theorems would be reduced to the merest trivialities. Is this observation any cause for sorrow? In the words of the great English mathematician, G. H. Hardy [**15**, p. 34]:

"*Reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game."

EXERCISES

1. Let BM and CN be external bisectors of the angles $B = 12°$ and $C = 132°$ of a special triangle ABC, each terminated at the opposite side. Without using trigonometric functions, compare the lengths of the angle bisectors. (O. Bottema†).

2. Where does our proof of Theorem 1.51 break down if we try to apply it to Bottema's triangle (in which nobody could deny that $B < C$)?

3. Use Exercise 7 of Section 1.3 to obtain a "direct" proof of the Steiner-Lehmus theorem.

1.6 The orthic triangle

A good deal can be learned from inspection of Figure 1.6A, which shows an acute-angled triangle ABC, its circumcenter O, its orthocenter H, and its orthic triangle DEF. Let us explain our reasons for

† See Archibald Henderson, *Scripta Mathematica* 21 (1956), pp. 309–310.

marking several angles with the same symbol α, meaning $90° - A$. First, since $\triangle OA'C$ is similar to the triangle JBC of Figure 1.1A, $\angle A'OC = A$. Thus the angles at the base of the isosceles triangle OBC are each $90° - A$. The right triangles ABE and ACF give us the same value for $\angle EBA$ and $\angle ACF$. The equality of these last two angles could also have been seen from the fact that, since $\angle BEC$ and $\angle BFC$ are right angles, the quadrilateral $BCEF$ is inscribable in a circle. Making analogous use of the quadrilaterals $BDHF$ and $CEHD$, we find that

$$\angle HDF = \angle HBF = \angle EBF = \angle ECF = \angle ECH = \angle EDH.$$

Thus HD bisects $\angle EDF$.

Similarly, HE bisects $\angle FED$, and HF bisects $\angle DFE$. A first interesting result, therefore, is the following: The altitudes of a triangle bisect the angles of its orthic triangle. Expressing it in another form that has a certain linguistic flavor to it:

THEOREM 1.61. *The orthocenter of an acute-angled triangle is the incenter of its orthic triangle.*

We have noticed in Figure 1.6A that $\angle HDF = \angle DBO$. Since HD is perpendicular to DB, FD must be perpendicular to OB. Similarly, DE is perpendicular to OC, and EF to OA.

EXERCISES

1. $\triangle AEF \sim \triangle DBF$, $\triangle DEC \sim \triangle ABC$ (Figure 1.6A).

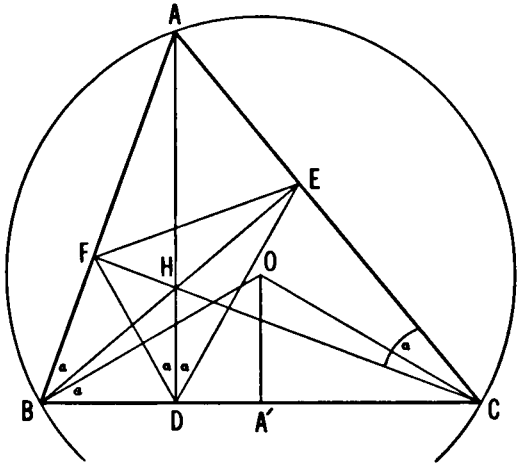


Figure 1.6A

2. Draw a new version of Figure 1.6A, with an obtuse angle at A. Which of the above conclusions have to be altered?

3. The orthocenter of an obtuse-angled triangle is an excenter of its orthic triangle.

4. $\angle HAO = |B - C|$.

1.7 The medial triangle and Euler line

The triangle formed by joining the midpoints of the sides of a given triangle will be called the *medial* triangle. In Figure 1.7A, $\triangle A'B'C'$ is the medial triangle of $\triangle ABC$. We have inserted the two medians AA' and BB' meeting at G, two altitudes of $\triangle ABC$ meeting at H, and two altitudes of $\triangle A'B'C'$ meeting at O. It is remarkable how much we can find out merely from an inspection of this figure.

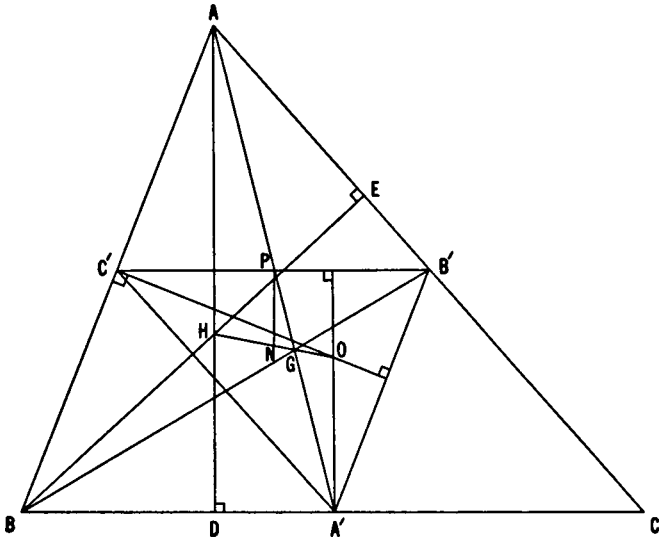


Figure 1.7A

First, $\triangle A'B'C'$ has its sides parallel to those of $\triangle ABC$, so the two triangles are similar. Next, $C'B' = \frac{1}{2}BC$, so the ratio between any two corresponding *line segments* (not merely corresponding sides) will be 1:2. In fact, the line segments $B'C'$, $C'A'$, $A'B'$ dissect $\triangle ABC$ into four congruent triangles.

Next, we see that $AC'A'B'$ is a parallelogram, so that AA' bisects $B'C'$. Therefore, the medians of $\triangle A'B'C'$ lie along the medians of

$\triangle ABC$, which means that both triangles have the same centroid, G. Incidentally, the midpoint P of $B'C'$ is also the midpoint of AA'

Now, the altitudes of $\triangle A'B'C'$ that we have drawn are the perpendicular bisectors of the sides AB and BC of $\triangle ABC$. We conclude that O, the *orthocenter* of $\triangle A'B'C'$, is at the same time the *circumcenter* of $\triangle ABC$.

Since H is the orthocenter of $\triangle ABC$ while O is the orthocenter of the similar triangle $A'B'C'$, $AH = 2OA'$. From Theorem 1.32, we recall that $AG = 2GA'$. Finally, since AD and OA' are both perpendicular to the side BC, they are parallel. Hence

$$\angle HAG = \angle OA'G, \qquad \triangle HAG \sim \triangle OA'G,$$

and

$$\angle AGH = \angle A'GO.$$

This shows that the points O, G, H are collinear, and $HG = 2GO$:

THEOREM 1.71. *The orthocenter, centroid and circumcenter of any triangle are collinear. The centroid divides the distance from the orthocenter to the circumcenter in the ratio* 2:1.

The line on which these three points lie is called the *Euler line* of the triangle.

Let us study Figure 1.7A more closely. We have marked the point N where the Euler line HO meets the line through P perpendicular to $B'C'$. The three lines AH, PN, $A'O$, all perpendicular to $B'C'$, are parallel. Since $AP = PA'$, they are evenly spaced: PN is midway between AH and $A'O$. Hence N is the midpoint of the segment HO.

We have conducted our discussions with respect to the side $B'C'$ of $\triangle A'B'C'$. If we apply the same reasoning to either of the other sides instead, the line segment HO remains fixed and is bisected by the perpendicular bisector of the new side. Since HO has just one midpoint, we can state that the perpendicular bisectors of all three sides of $\triangle A'B'C'$ will pass through the point N. In other words, N must be the circumcenter of $\triangle A'B'C'$.

To summarize, the circumcenter of the medial triangle lies at the midpoint of segment HO of the Euler line of the parent triangle. Also, since $\triangle A'B'C' \sim \triangle ABC$, the circumradius of the medial triangle equals half the circumradius of the parent triangle.

The name Euler appears so frequently and in so many branches of mathematics that a few words about him are in order. Leonhard Euler was born in 1707 in Basel, Switzerland. In 1727, he was invited to the St. Petersburg Academy in Russia. In 1741, he left for Berlin, to take the chair in mathematics at the Prussian Academy. He returned to St. Petersburg in 1766, and remained there until his death in 1783.

Euler was a tireless worker, his activities enriching every field of mathematics. Wherever one looks, there is either an Euler's theorem, an Euler's formula, or an Euler's method. Euler wrote 473 memoirs that were published during his lifetime, 200 that were published soon after, and 61 others that had to wait. Moreover, he did all this under a severe handicap, for he lost the sight of one eye in 1735, and the sight of the other in 1766. His skill in manipulation was remarkable, and his intuitive grasp of mathematics enormous. We shall meet his name again and again in our work.

EXERCISES

1. By drawing a new version of Figure 1.7A, based on Figure 1.1B instead of 1.1A, verify that our proof of Theorem 1.71 remains valid when $\triangle ABC$ has an obtuse angle.

2. $OH^2 = 9R^2 - a^2 - b^2 - c^2$.

3. $DA' = |b^2 - c^2|/2a$.

4. If $\triangle ABC$ has the special property that its Euler line is parallel to its side BC, then $\tan B \tan C = 3$.

1.8 The nine-point circle

To make things a bit easier, we remove some of the lines from Figure 1.7A and then add a few others; the result is Figure 1.8A. Let us see what we can read from this diagram, in which K, L, M are the midpoints of the segments AH, BH, CH of the three altitudes. Since BC is a common side of the two triangles ABC and HBC, whose other sides are bisected, respectively, by C', B' and L, M, both the segments $C'B'$ and LM are parallel to BC (and half as long). Similarly, since AH is a common side of the two triangles BAH and CAH, both the segments $C'L$ and $B'M$ are parallel to AH (and half as long). Hence $B'C'LM$ is a parallelogram. Since BC and AH are perpendicular, this parallelogram is a rectangle. Similarly, $A'B'KL$ is a rectangle (and so also $C'A'MK$). Hence $A'K$, $B'L$, $C'M$ are three diameters of a circle, as in Figure 1.8B.

Since $\angle A'DK$ is a right angle, this circle (on $A'K$ as diameter) passes through D. Similarly, it passes through E and F. To sum up:

THEOREM 1.81. *The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices to the orthocenter, all lie on the same circle, of radius* $\frac{1}{2}R$.

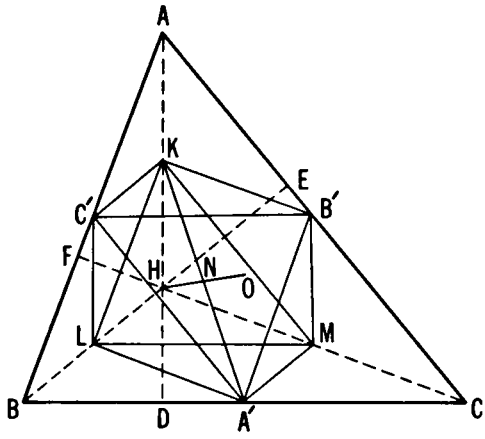


Figure 1.8A

Following J. V. Poncelet, we call this circle the *nine-point circle* of the triangle. Since the three points K, L, M are diametrically opposite to A', B', C', either of the two triangles KLM and $A'B'C'$ can be derived from the other by a half-turn (that is, a rotation through 180°) about the center of this circle. Clearly, this half-turn, which interchanges the two congruent triangles, must also interchange their orthocenters, H and O. Hence the center of the nine-point circle is the midpoint of HO, which we have already denoted by N in preparation for its role as the *nine-point center*. In other words:

THEOREM 1.82. *The center of the nine-point circle lies on the Euler line, midway between the orthocenter and the circumcenter.*

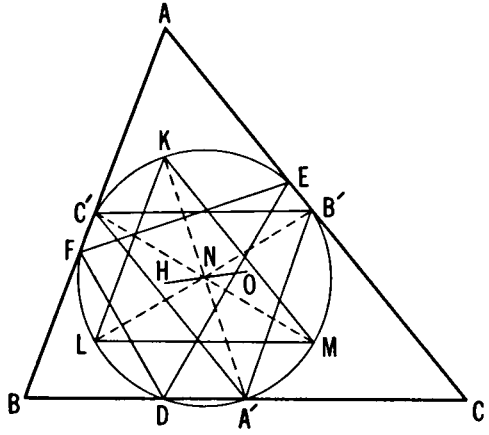


Figure 1.8B

The history of these two theorems is somewhat confused. A problem by B. Bevan that appeared in an English journal in 1804 seems to indicate that they were known then. They are sometimes mistakenly attributed to Euler, who proved, as early as 1765, that *the orthic triangle and the medial triangle have the same circumcircle.* In fact, continental writers often call the circle "the Euler circle". The first complete proof appears to be that of Poncelet, published in 1821. K. Feuerbach rediscovered Euler's partial result even later, and added a further property which is so remarkable that it has induced many authors to call the nine-point circle "the Feuerbach circle". Feuerbach's theorem (which we shall prove in Section 5.6) states that *the nine-point circle touches all the four tritangent circles.*

EXERCISES

1. The quadrilateral $AKA'O$ (Figure 1.8A) is a parallelogram.

2. In the nine-point circle (Figure 1.8B), the points K, L, M bisect the respective arcs EF, FD, DE.

3. The circumcircle of $\triangle ABC$ is the nine-point circle of $\triangle I_aI_bI_c$.

4. Let three congruent circles with one common point meet again in three points A, B, C. Then the common radius of the three given circles is equal to the circumradius of $\triangle ABC$, and their common point is its orthocenter.

5. The nine-point circle cuts the sides of the triangle at angles $|B - C|$, $|C - A|$, $|A - B|$.

1.9 Pedal triangles

The orthic triangle and the medial triangle are two instances of a more general type of associated triangle. Let P be any point inside a given triangle ABC, and let perpendiculars PA_1, PB_1, PC_1 be dropped to the three sides BC, CA, AB, as in Figure 1.9A. The feet of these perpendiculars are the vertices of a triangle $A_1B_1C_1$ which is called the *pedal triangle* of $\triangle ABC$ for the "pedal point" P. The restriction of P to interior positions can be relaxed if we agree to insist that (for a reason that will be explained in Section 2.5) P shall not lie on the circumcircle of $\triangle ABC$. Clearly, the orthic triangle or the medial triangle arises when P is the orthocenter or the circumcenter, respectively.

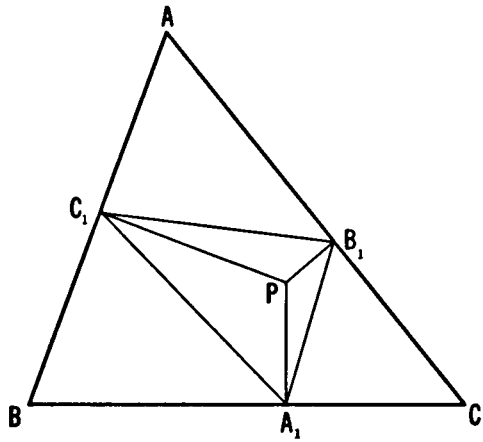


Figure 1.9A

Let us examine Figure 1.9A more closely. The right angles at B_1 and C_1 indicate that these points lie on the circle with diameter AP; in other words, P lies on the circumcircle of $\triangle AB_1C_1$. Applying the Law of Sines to this triangle and also to $\triangle ABC$ itself, we obtain

$$\frac{B_1C_1}{\sin A} = AP, \qquad \frac{a}{\sin A} = 2R,$$

whence

$$B_1C_1 = a\frac{AP}{2R}.$$

Similarly,

$$C_1A_1 = b\frac{BP}{2R} \quad \text{and} \quad A_1B_1 = c\frac{CP}{2R}.$$

We have thus proved:

THEOREM 1.91. *If the pedal point is distant x, y, z from the vertices of $\triangle ABC$, the pedal triangle has sides*

$$\frac{ax}{2R}, \quad \frac{by}{2R}, \quad \frac{cz}{2R}.$$

The case when $x = y = z = R$ is, of course, familiar.

An interesting exercise involving pedal triangles of pedal triangles is at the same time a delightful example of imagination in geometry. It seems to have first appeared when it was added, by the editor J. Neuberg,

to the sixth edition (1892) of John Casey's classic *A Sequel to the First Six Books of the Elements of Euclid.* In Figure 1.9B an interior point P has been used to determine $\triangle A_1B_1C_1$, the (first) pedal triangle of $\triangle ABC$. The same pedal point P has been used again to determine $\triangle A_2B_2C_2$, the pedal triangle of $\triangle A_1B_1C_1$, which we naturally call the "second pedal triangle" of $\triangle ABC$. A third operation yields $\triangle A_3B_3C_3$, the pedal triangle of $\triangle A_2B_2C_2$. The understanding is that, for this "third pedal triangle" also, we use the same pedal point P. In this terminology, Neuberg's discovery can be expressed thus:

THEOREM 1.92. *The third pedal triangle is similar to the original triangle.*

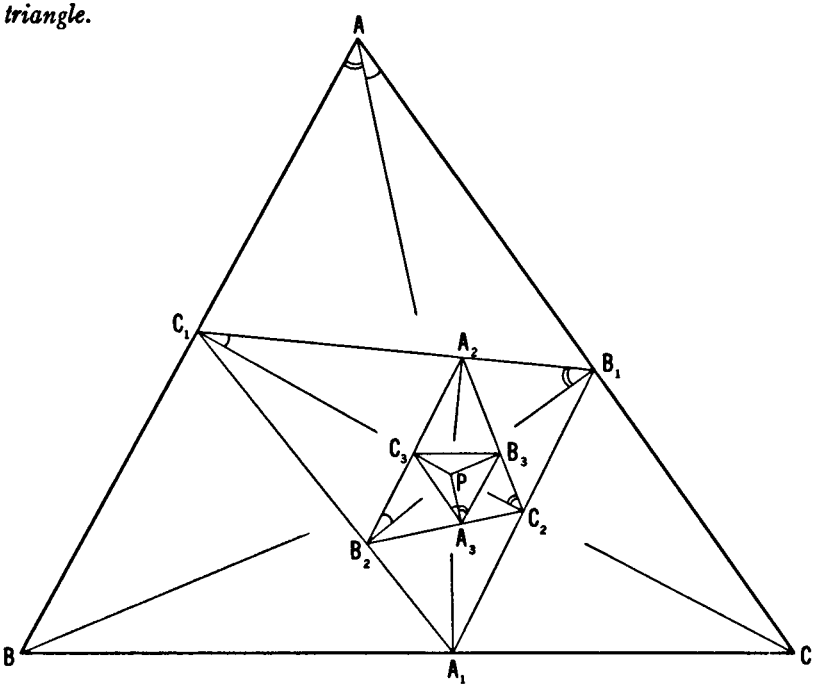


Figure 1.9B

The proof is surprisingly simple. The diagram practically gives it away, as soon as we have joined P to A. Since P lies on the circumcircles of all the triangles AB_1C_1, $A_2B_1C_2$, $A_3B_3C_2$, $A_2B_2C_1$, and $A_3B_2C_3$, we have

$$\angle C_1AP = \angle C_1B_1P = \angle A_2B_1P = \angle A_2C_2P$$
$$= \angle B_3C_2P = \angle B_3A_3P$$

and

$$\angle PAB_1 = \angle PC_1B_1 = \angle PC_1A_2 = \angle PB_2A_2$$
$$= \angle PB_2C_3 = \angle PA_3C_3.$$

In other words, the two parts into which AP divides $\angle A$ (marked in the diagram with a single arc and a double arc) have their equal counterparts at B_1 and C_1, again at C_2 and B_2, and finally both at A_3. Hence $\triangle ABC$ and $\triangle A_3B_3C_3$ have equal angles at A and A_3. Similarly, they have equal angles at B and B_3. Thus the theorem is proved.

It is interesting to follow in the diagram the "parade of angles" from position A to position A_3: as neat as the maneuvers of a drill team.

This property of continued pedals has been generalized by B. M. Stewart (*Am. Math. Monthly*, vol. 47, Aug.-Sept. 1940, pp. 462-466). He finds that *the nth pedal n-gon of any n-gon is similar to the original n-gon.* It is instructive to try this for the fourth pedal quadrilateral of a quadrilateral.

At this point let us pause in our investigations. We have done part of what we set out to do: beginning with well-known data, we have developed a few simple but significant facts. There are many problems that lend themselves to solution by the methods described here. Some of them are well-known posers that the reader may have seen before. We bring this chapter to a close by presenting five of these hardy perennials.

EXERCISES

1. If a cevian AQ of an equilateral triangle ABC is extended to meet the circumcircle at P, then

$$\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PQ}.$$

2. If an isosceles triangle PAB, with equal angles 15° at the ends of its base AB, is drawn inside a square $ABCD$, as in Figure 1.9C, then the points P, C, D are the vertices of an equilateral triangle.

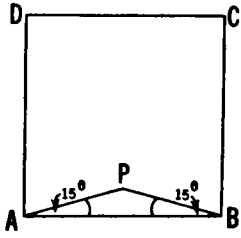


Figure 1.9C

3. If lines PB and PD, outside a parallelogram $ABCD$, make equal angles with the sides BC and DC, respectively, as in Figure 1.9D, then $\angle CPB = \angle DPA$. (Of course, this is a plane figure, not three dimensional!)

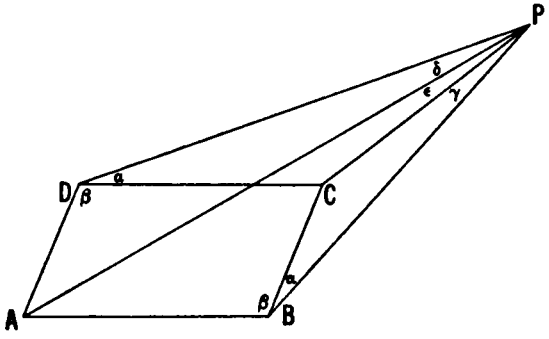


Figure 1.9D

4. Let ABC be an isosceles triangle with equal angles $80°$ at B and C. Cevians BD and CE divide $\angle B$ and $\angle C$ into

$$60° + 20° \quad \text{and} \quad 30° + 50°,$$

as in Figure 1.9E. Find $\angle EDB$.

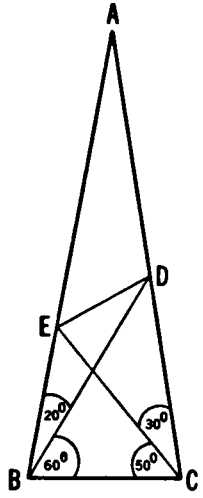


Figure 1.9E

5. If two lines through one vertex of an equilateral triangle divide the semicircle drawn outward on the opposite side into three equal arcs, these same lines divide the side itself into three equal line segments.

CHAPTER 2

Some Properties of Circles

> Although the Greeks worked fruitfully, not only in geometry but also in the most varied fields of mathematics, nevertheless we today have gone beyond them everywhere and certainly also in geometry.
>
> *F. Klein*

The circle has been held in highest esteem through the ages. Its perfect form has affected philosophers and astronomers alike. Until Kepler derived his laws, the thought that planets might move in anything but circular paths was unthinkable. Nowadays, the words "square", "line", and the like sometimes have derogatory connotations, but the circle—never. Cleared of superstitious nonsense and pseudo-science, it still stands out, as estimable as ever.

Limitations of space make it impossible for us to present more than a few of the most interesting properties developed since Euclid of the circle and its relation to triangles and other polygons.

2.1 The power of a point with respect to a circle

We begin our investigations by recalling two of Euclid's theorems: III.35, about the product of the parts into which two chords of a circle divide each other (that is, in the notation of Figure 2.1A, $PA \times PA' = PB \times PB'$), and III.36, comparing a secant and a tangent drawn from the same point P outside the circle (in Figure 2.1B, $PA \times PA' = PT^2$). If we agree to regard a tangent as the limiting form of a secant, we can combine these results as follows:

THEOREM 2.11. *If two lines through a point P meet a circle at points A, A' (possibly coincident) and B, B' (possibly coincident), respectively, then* $PA \times PA' = PB \times PB'$.

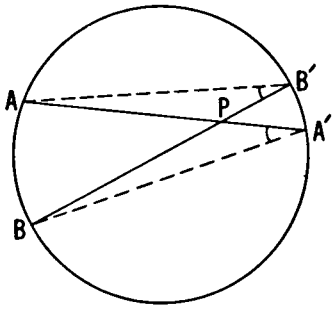


Figure 2.1A

For a proof we merely have to observe that the similar triangles PAB' and PBA' (with a common angle at P) yield

$$\frac{PA}{PB'} = \frac{PB}{PA'}.$$

In Figure 2.1B, we can equally well use the similar triangles PAT and PTA' to obtain

$$\frac{PA}{PT} = \frac{PT}{PA'},$$

and then say $PA \times PA' = PT^2 = PB \times PB'$

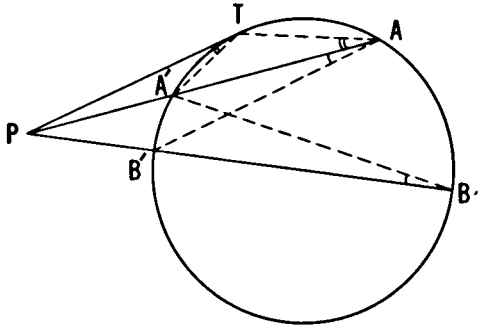


Figure 2.1B

Let R denote the radius of the circle, and d the distance from P to the center. By taking BB' to be the diameter through P (with B farther from P than B'), we see that, if P is inside the circle (as in Figure 2.1A),

$$AP \times PA' = BP \times PB' = (R + d)(R - d) = R^2 - d^2,$$

and if P is outside (as in Figure 2.1B),

$$PA \times PA' = PB \times PB' = (d + R)(d - R) = d^2 - R^2.$$

The equation

$$AP \times PA' = R^2 - d^2$$

provides a quick proof of a formula due to Euler:

THEOREM 2.12. *Let O and I be the circumcenter and incenter, respectively, of a triangle with circumradius R and inradius r; let d be the distance OI. Then*

$$d^2 = R^2 - 2rR. \tag{2.12}$$

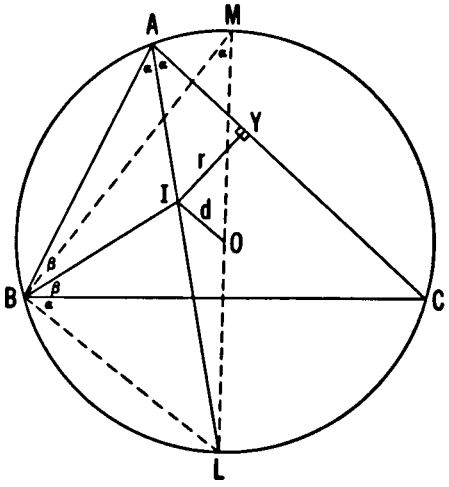


Figure 2.1C

Figure 2.1C shows the internal bisector of $\angle A$ extended to meet the circumcircle at L, the midpoint of the arc BC not containing A. LM is the diameter perpendicular to BC. Writing, for convenience, $\alpha = \frac{1}{2}A$ and $\beta = \frac{1}{2}B$, we notice that

$$\angle BML = \angle BAL = \alpha, \quad \text{and} \quad \angle LBC = \angle LAC = \alpha.$$

Since the exterior angle of $\triangle ABI$ at I is

$$\angle BIL = \alpha + \beta = \angle LBI,$$

$\triangle LBI$ is isosceles: $LI = LB$. Thus

$$\begin{aligned} R^2 - d^2 &= LI \times IA = LB \times IA \\ &= LM \frac{LB/LM}{IY/IA} IY = LM \frac{\sin \alpha}{\sin \alpha} IY \\ &= LM \times IY = 2Rr, \end{aligned}$$

that is, $d^2 = R^2 - 2rR$, as we wished to prove.

For any circle of radius R and any point P distant d from the center, we call

$$d^2 - R^2$$

the *power* of P with respect to the circle. It is clearly positive when P is outside, zero when P lies on the circumference, and negative when P is inside. For the first of these cases we have already obtained the alternative expression

$$PA \times PA',$$

where A and A' are any two points on the circle, collinear with P (as in Theorem 2.11). This expression for the power of a point P remains valid for all positions of P if we agree to adopt Newton's idea of *directed* line segments: a kind of one-dimensional vector algebra in which

$$AP = -PA.$$

The product (or quotient) of two directed segments on one line is regarded as being positive or negative according as the directions agree or disagree. With this convention, the equation

$$d^2 - R^2 = PA \times PA'$$

holds universally. If P is inside the circle,

$$d^2 - R^2 = -(R^2 - d^2) = -AP \times PA' = PA \times PA';$$

and if P is on the circumference, either A or A' coincides with P, so that one of the segments has length zero. In fact, after observing that the product $PA \times PA'$ has the same value for every secant (or chord) through P, we could have used this value as a *definition* for the power of P with respect to the circle.

The word *power* was first used in this sense by Jacob Steiner, whose name has already appeared in Chapter 1.

EXERCISES

1. What is the (algebraically) smallest possible value that the power of a point can have with respect to a circle of given radius R? Which point has this critical power?

2. What is the locus of points of constant power (greater than $-R^2$) with respect to a given circle?

3. If the power of a point has the positive value t^2, interpret the length t geometrically.

4. If PT and PU are tangents from P to two concentric circles, with T

on the smaller, and if the segment PT meets the larger circle at Q, then $PT^2 - PU^2 = QT^2$.

5. The circumradius of a triangle is at least twice the inradius.

6. Express (in terms of r and R) the power of the incenter with respect to the circumcircle.

7. The notation of directed segments enables us to express Stewart's theorem (Exercise 4 of Section 1.2) in the following symmetrical form [5, p. 152]: *If P, A, B, C are four points of which the last three are collinear, then* $PA^2 \times BC + PB^2 \times CA + PC^2 \times AB + BC \times CA \times AB = 0$.

8. A line through the centroid G of ΔABC intersects the sides of the triangle at points X, Y, Z. Using the concept of directed line segments, prove that

$$\frac{1}{GX} + \frac{1}{GY} + \frac{1}{GZ} = 0.$$

9. How far away is the horizon as seen from the top of a mountain one mile high? (Assume the earth to be a sphere of diameter 7920 miles.)

2.2 The radical axis of two circles

The following anecdote was related by E. T. Bell [3, p. 48]. Young Princess Elisabeth, exiled from Bohemia, had successfully attacked a problem in elementary geometry by using coordinates. As Bell states it, "The problem is a fine specimen of the sort that are *not* adapted to the crude brute force of elementary Cartesian geometry." Her teacher was René Descartes (after whom Cartesian coordinates were named†). His reaction was that "he would not undertake to carry out her solution . . . in a month."

The lesson is clear: a solution that is possible in a certain manner may still not be the best or most economical one. At any rate, here is one theorem for which an analytic proof, without being any more difficult than the usual synthetic proof [6, p. 86], has some interesting repercussions:

THEOREM 2.21. *The locus of all points whose powers with respect to two nonconcentric circles are equal is a line perpendicular to the line of centers of the two circles.*

† There are some who claim that it was Pierre Fermat (1601–1665) who actually invented analytic geometry. Their contention is that he gave the essential idea to Descartes in a letter.

In terms of rectangular Cartesian coordinates, the square of the distance d between any two points (x, y) and (a, b) is

$$(x - a)^2 + (y - b)^2.$$

Therefore the power of (x, y) with respect to the circle with center (a, b) and radius r is

$$d^2 - r^2 = (x - a)^2 + (y - b)^2 - r^2.$$

In particular, the circle itself, being the locus of points (x, y) of power zero, has the equation

$$(x - a)^2 + (y - b)^2 - r^2 = 0. \tag{2.22}$$

The same equation, in the form $(x - a)^2 + (y - b)^2 = r^2$, expresses the circle as the locus of points whose distances from (a, b) have the constant value r.

When this circle is expressed in the form

$$x^2 + y^2 - 2ax - 2by + c = 0 \tag{2.23}$$

(where $c = a^2 + b^2 - r^2$), the power of an arbitrary point (x, y) is again expressed by the left side of the equation, namely

$$x^2 + y^2 - 2ax - 2by + c.$$

Another circle having the same center (a, b) but a different radius has an equation of the same form with a different c, and any circle having a different center has an equation of the form

$$x^2 + y^2 - 2a'x - 2b'y + c' = 0, \tag{2.24}$$

where either $a' \neq a$ or $b' \neq b$ or both. We are thus free to use the equations (2.23) and (2.24) for the two non-concentric circles mentioned in Theorem 2.21. The locus of all points (x, y) whose powers with respect to these two circles are equal is

$$x^2 + y^2 - 2ax - 2by + c = x^2 + y^2 - 2a'x - 2b'y + c'.$$

Since $x^2 + y^2$ cancels, this locus is the *line*

$$(a' - a)x + (b' - b)y = \tfrac{1}{2}(c' - c).$$

By choosing our frame of reference so that the x-axis joins the two centers, we may express the two circles in the simpler form

$$x^2 + y^2 - 2ax + c = 0, \qquad x^2 + y^2 - 2a'x + c' = 0, \tag{2.25}$$

where $a' \neq a$. Then the locus becomes

$$x = \frac{c' - c}{2(a' - a)}.$$

This line, being parallel to the y-axis, is *perpendicular* to the x-axis,

which is the join of centers. Since the line can be defined geometrically in terms of the circles (as containing all points of equal power), we could have taken it to be the y-axis itself, as in Figure 2.2A. Thus any two non-concentric circles can be expressed in the still simpler form

$$(2.26) \qquad x^2 + y^2 - 2ax + c = 0, \qquad x^2 + y^2 - 2a'x + c = 0.$$

Now the locus is $x = 0$. Conversely, every point $(0, y)$ on the line $x = 0$ has the *same* power $y^2 + c$ with respect to both circles.

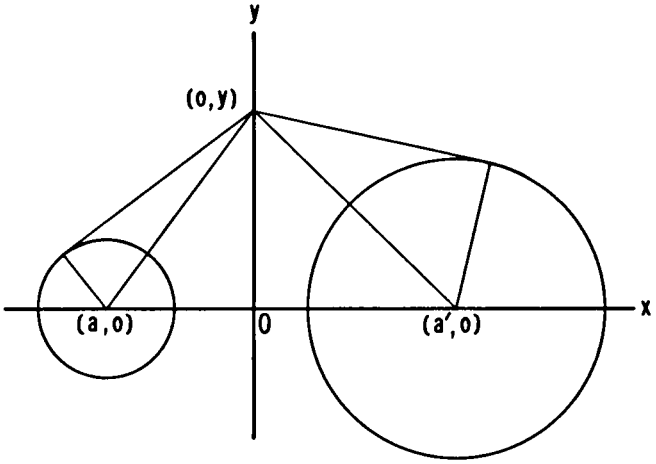


Figure 2.2A

This remark completes the proof. Of course, we could have shortened it by expressing the two circles immediately in the form (2.25); but then we would have missed the beautiful lemma that, for any circle expressed in the standard form (2.23), *the power of the general point* (x, y) *is equal to the expression on the left side of the equation.*

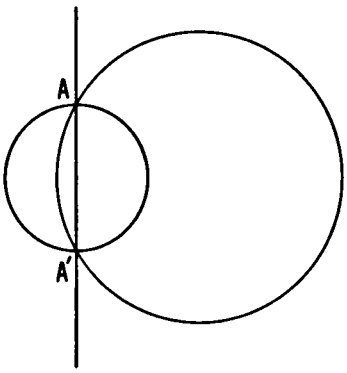


Figure 2.2B

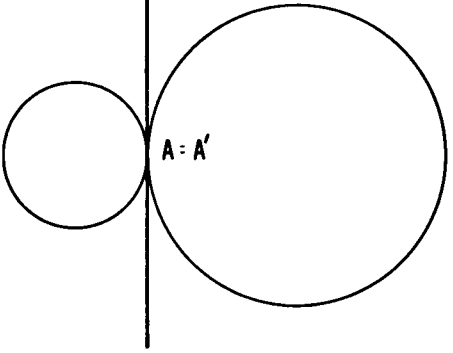


Figure 2.2C

The locus of points of equal power with respect to two non-concentric circles is called their *radical axis*. In the special case when the two circles intersect at two points A and A' (Figure 2.2B), each of these points has zero power for both circles, and therefore the radical axis is simply the line AA'. Similarly, when the two circles touch each other (Figure 2.2C), their radical axis is their common tangent at their point of contact.

EXERCISES

1. What is the locus of all points from which the tangents to two given circles have equal lengths?

2. When the distance between the centers of two circles is greater than the sum of the radii, the circles have four common tangents. The midpoints of these four line segments are collinear.

3. Let PAB, AQB, ABR, $P'BA$, $BQ'A$, BAR' be six *similar* triangles all on the same side of their common side AB. (Three of them are shown in Figure 2.2D; the rest can be derived by reflection† in the perpendicular bisector of the segment AB.) Those vertices of the triangles that do not lie on AB (namely, P, Q, R, P', Q', R') all lie on one circle. *Hint*: Compare the powers of A and B with respect to the circle PQR.

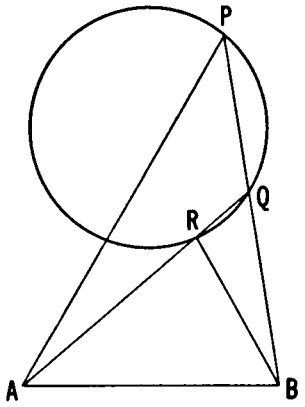


Figure 2.2D

4. Given a and b, for what values of c does the equation 2.23 represent a circle?

5. Describe a construction for the radical axis of two given non-concentric circles: a construction that remains valid when one circle encloses the other.

† The operation of reflection is useful in solving many geometric problems. See, for instance, Yaglom [29].

2.3 Coaxal circles

The two circles (2.26) (which may be any two non-concentric circles) are members of an infinite family, represented by the equation

$$x^2 + y^2 - 2ax + c = 0,$$

where c is fixed while a varies over the whole range of real values (except, if c is positive, the values between $\pm\sqrt{c}$). This family is called a *pencil of coaxal circles*, because every two of its members have the same line of centers and the same radical axis. If c is negative, every member of the family meets the y-axis at the same two points $(0, \pm\sqrt{-c})$, and the pencil consists simply of *all the circles through these two points*. Similarly, if $c = 0$, the pencil consists of all the circles that touch the y-axis at the origin. The case when c is positive is illustrated in Figure 2.3A.

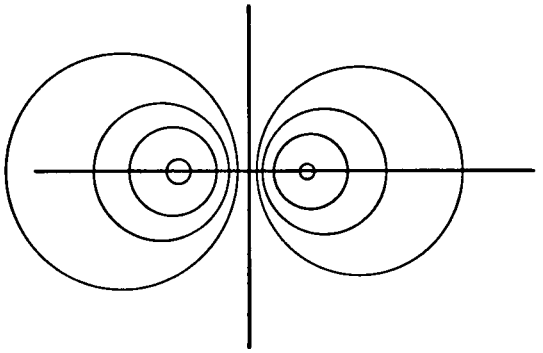


Figure 2.3A

If *three* non-coaxal circles are such that no two are concentric, we can take them in pairs and thus find three radical axes. Any point that has the same power for all three circles must lie on all three of these lines. Conversely, any point of intersection of two of the three radical axes, having the same power for all three circles, must lie on the third line as well. If two of the axes are parallel, then all three must be parallel. In particular:

THEOREM 2.31. *If the centers of three circles form a triangle, there is just one point whose powers with respect to the three circles are all equal.*

This common point of the three radical axes is called the *radical center* of the three circles.

EXERCISES

1. Two circles are in contact internally at a point T. Let the chord AB of

the larger circle be tangent to the smaller circle at a point P. Then the line TP bisects $\angle ATB$.

2. If three non-intersecting circles have radical center O, the points of contact of the six tangents from O to the circles all lie on one circle.

2.4 More on the altitudes and orthocenter of a triangle

The circumcircle of a triangle, already encountered in the previous chapters, deserves further examination. Figure 2.4A shows the circumcircle ABC with center O, diameter AA_0 through A, and radius $OL = R$ perpendicular to BC. We see also the altitude $AD = h_a$. The equal angles at B and A_0 make $\triangle ABD \sim \triangle AA_0C$, so that

$$\frac{h_a}{c} = \frac{b}{AA_0}$$

and

$$h_a = \frac{bc}{2R}. \tag{2.41}$$

Subtracting from $\angle BAC$ the two equal angles

$$\angle A_0AC = \angle BAD = 90° - B,$$

we are left with

$$\begin{aligned}\angle DAA_0 &= A - 2(90° - B) = A + 2B - (A + B + C)\\ &= B - C.\end{aligned}$$

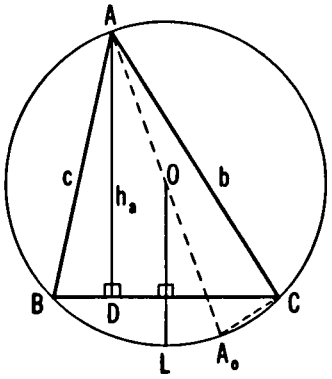


Figure 2.4A

This expression for $\angle DAA_0 = \angle DAO$ has been made with reference to the figure in which $B > C$. If instead we had taken $B < C$, the equal angles A_0AC and BAD would have overlapped, with the result that $\angle DAO = C - B$. We can include both cases by writing

$$\angle DAO = |B - C|. \tag{2.42}$$

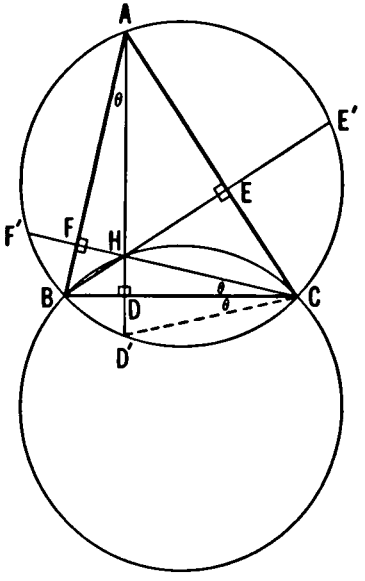


Figure 2.4B

Figure 2.4B shows the three altitudes AD, BE, CF extended to meet the circumcircle at D', E', F'. Of course, H is the orthocenter. Now $\angle DAB = \angle FCB$, both being complements of the angle B. This explains our use of the same symbol θ for both. Also $\angle BCD' = \angle BAD'$, so we have labeled $\angle BCD'$ accordingly. The congruent right-angled triangles CDH and CDD' show us that

$$HD = DD'. \tag{2.43}$$

Similarly, $HE = EE'$ and $HF = FF'$.

Since the circle with diameter AB passes through D and E, Theorem 2.11 tells us that $HA \times HD = HB \times HE$. Similarly $HB \times HE = HC \times HF$. Hence

$$HA \times HD = HB \times HE = HC \times HF. \tag{2.44}$$

If X, Y, Z are any points on the respective sides BC, CA, AB, circles constructed on the cevians AX, BY, CZ as diameters will pass through the feet of the altitudes: D, E, F, respectively. (The second and third circles are shown in Figure 2.4C.) The three expres-

sions equated in (2.44) are the powers of H with respect to these three circles. Hence H is the radical center of the circles, and we have proved two interesting theorems that have appeared at various times as puzzlers:

THEOREM 2.45. *If circles are constructed on two cevians as diameters, their radical axis passes through the orthocenter H of the triangle.*

THEOREM 2.46. *For any three non-coaxal circles having cevians for diameters, H is their radical center.*

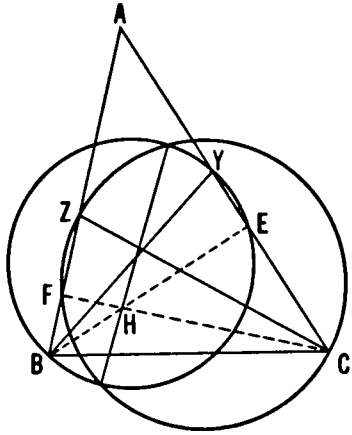


Figure 2.4C

Alternatively, the same results can be obtained by means of the following simple considerations. If AD is the altitude from A, the pencil of coaxal circles through A and D may be described as the circles having cevians through A as diameters. Two of these cevians are the sides AB and AC. Thus the circles on BC, CA, AB as diameters have the altitudes for their radical axes in pairs, and H for their radical center. (In this manner, the concurrence of the altitudes is seen to be a special case of Theorem 2.31). It follows that H has the same power for *all* circles having cevians for diameters.

Notice the word "non-coaxal" in the statement of Theorem 2.46. This implies that the three cevians are not all drawn from the same vertex of $\triangle ABC$. We shall see in our next theorem that it implies slightly more!

Several amusing problems can be derived from Theorem 2.46 (as applied to cevians AX, BY, CZ) by introducing non-essentials. Although the three cevians need not be concurrent, it makes for greater confusion to let them be concurrent. Thus, we might ask: if circles are constructed on the medians (or altitudes, or angle bisectors) as diameters, prove that their radical center is the orthocenter of the triangle.

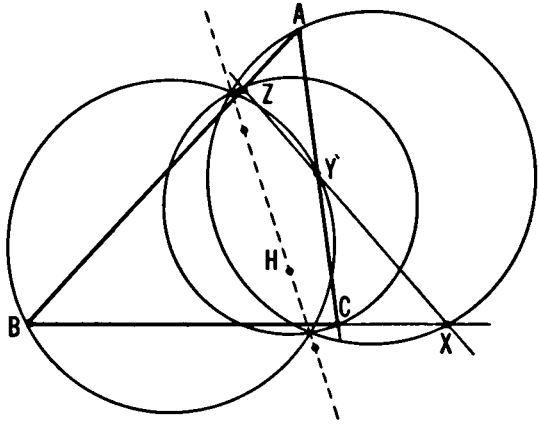


Figure 2.4D

The most interesting case of non-concurrent cevians arises when X, Y, Z are *collinear* points on the lines BC, CA, AB (extended if necessary), as in Figure 2.4D; for then we can equally well say that X, B, C are collinear points on the sides of $\triangle AYZ$, or that Y, C, A are collinear points on the sides of $\triangle BZX$, or that Z, A, B are collinear points on the sides of $\triangle CXY$. Hence the circles on AX, BY, CZ as diameters are so situated that their radical axes pass through H and also (for the same reason) through the orthocenters of the other three triangles. Since these four orthocenters are obviously distinct, the radical axes must coincide, and we have proved

THEOREM 2.47. *If four lines meet one another at six points A, B, C, X, Y, Z, so that the sets of collinear points are XBC, YCA, ZAB, XYZ, then the circles on AX, BY, CZ as diameters are coaxal, and the orthocenters of the four triangles AYZ, BZX, CXY, ABC are collinear.*

Another property of a triangle and its altitudes is illustrated in Figure 1.3C. If we inspect the diagram carefully, we reach the conclusion that, just as H is the orthocenter of $\triangle ABC$, A *is the orthocenter of* $\triangle HBC$, and, for the same reason, B is the orthocenter of $\triangle HAC$, and C is the orthocenter of $\triangle HAB$. This configuration $ABCH$ is known as an *orthocentric quadrangle*, and has a number of interesting properties. We merely examine one of them, namely: *If $ABCH$ is an orthocentric quadrangle, the circumcircles of the four triangles formed by taking any three of the vertices have equal radii.*

The simplest proof makes use of the equation (2.43) and Figure 2.4B. In this figure, $\triangle HBC$ and $\triangle D'BC$ are congruent, so they must have congruent circumcircles. Hence the circumcircle of $\triangle D'BC$ (or $\triangle ABC$) is congruent to the circumcircle of $\triangle HBC$, and similarly for the other triangles.

EXERCISES

1. The points where the extended altitudes meet the circumcircle form a triangle similar to the orthic triangle.

2. The internal angle bisectors of $\triangle ABC$ are extended to meet the circumcircle at points L, M, N, respectively. Find the angles of $\triangle LMN$ in terms of the angles A, B and C.

2.5 Simson lines

If perpendiculars are dropped onto the sides of a triangle ABC from a point P, the feet of these perpendiculars usually form the vertices of a triangle $A_1B_1C_1$ (the *pedal triangle* discussed in Section 1.9). Let us now examine the exceptional case where the point P lies on the circumcircle, as in Figure 2.5A. To be definite, we have taken P to lie on the arc CA that does not contain B, between A and the point diametrically opposite to B. All other cases can be derived by re-naming A, B, C. Because of the right angles at A_1, B_1 and C_1, P lies also on the circumcircles of triangles A_1BC_1, A_1B_1C and AB_1C_1. Therefore

$$\angle APC = 180° - B = \angle C_1PA_1$$

and, subtracting $\angle APA_1$, we deduce

$$\angle A_1PC = \angle C_1PA.$$

But since points A_1, C, P, B_1 lie on a circle,

$$\angle A_1PC = \angle A_1B_1C,$$

and since points A, B_1, P, C_1 lie on a circle,

$$\angle C_1PA = \angle C_1B_1A.$$

Thus

$$\angle A_1B_1C = \angle C_1B_1A,$$

so that the points A_1, B_1, C_1 are collinear; the pedal triangle is "degenerate.".

Conversely, if a point P is so situated that the pedal triangle of $\triangle ABC$ is degenerate, P must evidently lie in the region of the plane that is inside one angle of $\triangle ABC$ and beyond the opposite side. By re-naming the vertices if necessary, we can assume that this "one angle" is B, and that C_1 lies on the extension of the side BA beyond A, as in Figure 2.5A. We can then reverse the steps in the above discussion of angles and conclude that P lies on the circumcircle. Hence

THEOREM 2.51. *The feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumcircle.*

The line containing the feet is known as the *Simson line* (or sometimes just the *simson*) of the point with respect to the triangle. Robert Simson (1687–1768) made several contributions to both geometry and arithmetic. For instance, it was he who discovered that, if f_n is the nth term of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, $\cdots$, then $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$ [6, pp. 165–168]. The "simson" was attributed to him because it seemed to be typical of his geometrical ideas. However, historians have searched through his works for it in vain. Actually it was discovered in 1797 by William Wallace.

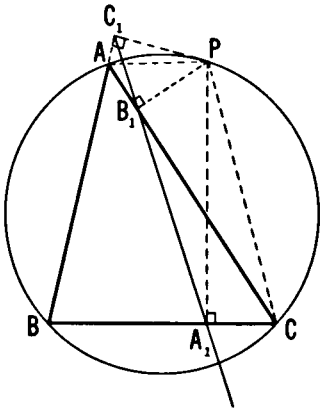


Figure 2.5A

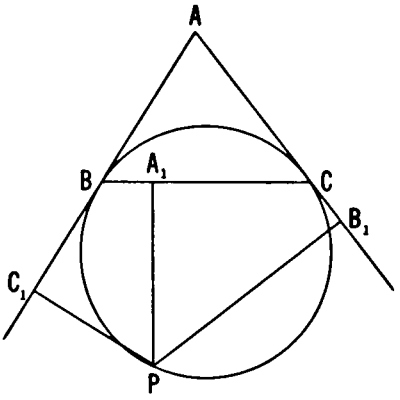


Figure 2.5B

EXERCISES

1. Does our proof of Theorem 2.51 require any modification when $\triangle ABC$ has an obtuse angle?

2. What point on the circle has CA as its Simson line?

3. Are there any points that lie on their own Simson lines? What lines are these?

4. The tangents at two points B and C on a circle meet at A. Let $A_1B_1C_1$ be the pedal triangle of the isosceles triangle ABC for an arbitrary point P on the circle, as in Figure 2.5B. Then

$$PA_1^2 = PB_1 \times PC_1.$$

2.6 Ptolemy's theorem and its extension

The concept of the Simson line can be used to derive a very useful theorem, as follows. Let us examine Figure 2.5A again. Although the "pedal triangle" $A_1B_1C_1$ is degenerate, the lengths of its "sides" are still given by Theorem 1.91:

$$B_1C_1 = \frac{aAP}{2R}, \qquad A_1C_1 = \frac{bBP}{2R}, \qquad A_1B_1 = \frac{cCP}{2R}.$$

Since $A_1B_1 + B_1C_1 = A_1C_1$, we deduce $cCP + aAP = bBP$, that is

$$AB \times CP + BC \times AP = AC \times BP.$$

Since $ABCP$ is a cyclic quadrilateral, we have thus proved Ptolemy's theorem:

THEOREM 2.61. *If a quadrilateral is inscribed in a circle, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.*

Ptolemy's theorem has a converse that can be strengthened by observing that, for any location of B_1 other than on the segment A_1C_1, the equation $A_1B_1 + B_1C_1 = A_1C_1$ has to be replaced by the "triangle inequality"

$$A_1B_1 + B_1C_1 > A_1C_1,$$

which yields

$$AB \times CP + BC \times AP > AC \times BP.$$

Hence

THEOREM 2.62. *If ABC is a triangle and P is not on the arc CA of its circumcircle, then*

$$AB \times CP + BC \times AP > AC \times BP.$$

EXERCISES

1. Let P be any point in the plane of an equilateral triangle ABC. Then $PC + PA = PB$ or $PC + PA > PB$ according as P does or does not lie on the arc CA of the circumcircle. [For an interesting application of this result, see **23**, pp. 11–12.]

2. If a point P lies on the arc CD of the circumcircle of a square $ABCD$, then $PA(PA + PC) = PB(PB + PD)$.

3. If a circle cuts two sides and a diagonal of a parallelogram $ABCD$ at points P, R, Q as shown in Figure 2.6A, then

$$AP \times AB + AR \times AD = AQ \times AC.$$

Hint: Apply Theorem 2.61 to the quadrilateral $PQRA$ and then replace the sides of $\triangle PQR$ by the corresponding sides of the similar triangle CBA.

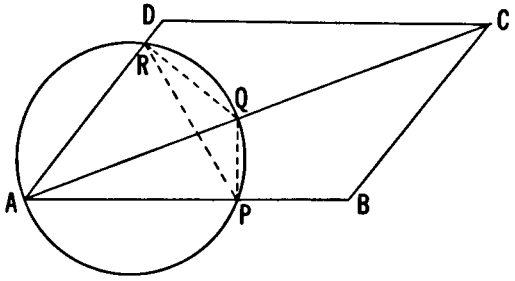


Figure 2.6A

2.7 More on Simson lines

The Simson line has many interesting properties, and it may be worthwhile to investigate some of them. Let us begin by examining Figure 2.7A, which is the same as Figure 2.5A except that the perpendicular PA_1 has been extended to meet the circumcircle at U, and the line AU has been drawn.

The cyclic quadrilaterals $PAUC$ and PB_1A_1C tell us that

$$\angle PUA = \angle PCA = \angle PCB_1 = \angle PA_1B_1.$$

Therefore *the line* AU *is parallel to the Simson line* A_1B_1.

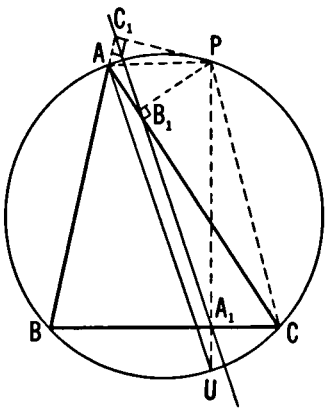


Figure 2.7A

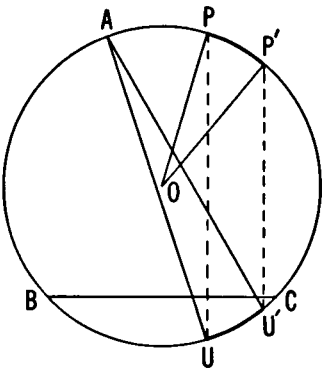


Figure 2.7B

Let us now compare the Simson line of P with the Simson line of another point P' (also, of course, on the circumcircle). The angle between these two Simson lines is simply the angle UAU' between the lines AU and AU' which are parallel to them (Figure 2.7B). The two chords PU and $P'U'$, both perpendicular to BC, are parallel to each other, and cut off equal arcs PP' and UU'. Thus

$$\angle UAU' = \tfrac{1}{2}\angle UOU' = \tfrac{1}{2}\angle POP'$$

or, if we distinguish between positive and negative angles,

$$\angle UAU' = \tfrac{1}{2}\angle UOU' = -\tfrac{1}{2}\angle POP'.$$

We have thus proved:

THEOREM 2.71. *The angle between the Simson lines of two points P and P' on the circumcircle is half the angular measure of the arc $P'P$.*

If we imagine P to run steadily round the circumcircle, the line AU will rotate steadily about A at half the angular velocity in the opposite sense, so as to reverse its direction by the time P has described the whole circumference. Meanwhile, the Simson line will turn in a corresponding manner about a continuously changing center of rotation. In fact, the Simson line envelops a beautifully symmetrical curve called a deltoid or "Steiner's hypocycloid" [20]. The motion is demonstrated very clearly in the film *Simson Line* by T. J. Fletcher.

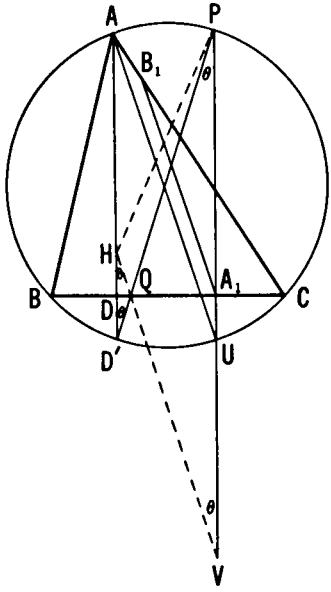


Figure 2.7C

To continue our investigation, let us now examine Figure 2.7C, which is a combination of Figures 2.4B and 2.7A with the extra lines HP, $D'P$ (meeting BC at Q) and HQ (extended to meet PU at V). Since both HD' and PV are perpendicular to BC, equation (2.43) shows that the triangles QHD' and QPV are isosceles. In other words, HV is the image of $D'P$ by reflection in BC. Since

$$\angle D'HV = \angle PVH = \angle D'PU = \angle D'AU,$$

the line HV is parallel to AU, which we have already shown to be parallel to the Simson line of P.

Finally, we observe that, in $\triangle PHV$, the Simson line A_1B_1 is parallel to the side HV and bisects the side PV (at A_1). Hence it must also bisect the remaining side PH:

THEOREM 2.72. *The Simson line of a point (on the circumcircle) bisects the segment joining that point to the orthocenter.*

This has been merely an introduction to the topic of Simson lines. They have many other properties which we must regretfully leave to other sources.

EXERCISES

1. The Simson lines of diametrically opposite points on the circumcircle are perpendicular to each other and meet on the nine-point circle.

2. Let ABC be an equilateral triangle inscribed in a circle with center O, and let P be any point on the circle. Then the Simson line of P bisects the radius OP.

2.8 The Butterfly

The Butterfly theorem has been around for quite a while. We state it as follows (see Figure 2.8A):

THEOREM 2.81. *Through the midpoint M of a chord PQ of a circle, any other chords AB and CD are drawn; chords AD and BC meet PQ at points X and Y. Then M is the midpoint of XY.*

For this theorem numerous proofs have been developed, varying in length and difficulty. Three have been received from Dr. Zoll of Newark State College. He mentioned that one of these was submitted in 1815

by W. G. Horner, discoverer of Horner's Method for approximating the roots of a polynomial equation. (According to E. T. Bell, Horner's Method was anticipated by a Chinese.) For another proof, see R. Johnson [17, p. 78]. The shortest proof employs projective geometry [7, pp. 78, 144]. The one presented here, though not very short, is simple and easy to remember.

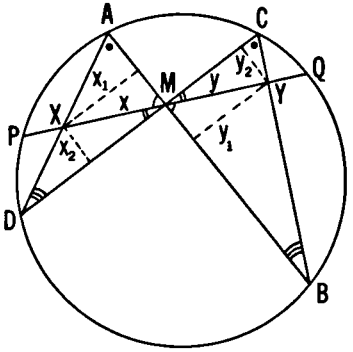


Figure 2.8A

We begin by dropping perpendiculars x_1 and y_1 from X and Y to AB, x_2 and y_2 from X and Y to CD. Writing for convenience $a = PM = MQ$, $x = XM$, $y = MY$, we observe that the pairs of similar triangles Mx_1 and My_1, Mx_2 and My_2, Ax_1 and Cy_2, Dx_2 and By_1 yield

$$\frac{x}{y} = \frac{x_1}{y_1}, \quad \frac{x}{y} = \frac{x_2}{y_2}, \quad \frac{x_1}{y_2} = \frac{AX}{CY}, \quad \frac{x_2}{y_1} = \frac{XD}{YB},$$

whence

$$\frac{x^2}{y^2} = \frac{x_1}{y_1}\frac{x_2}{y_2} = \frac{x_1}{y_2}\frac{x_2}{y_1} = \frac{AX \times XD}{CY \times YB} = \frac{PX \times XQ}{PY \times YQ}$$

$$= \frac{(a-x)(a+x)}{(a+y)(a-y)} = \frac{a^2-x^2}{a^2-y^2} = \frac{a^2}{a^2} = 1,$$

and $x = y$, as we wished to prove.

EXERCISES

1. In Figure 2.8A the lines AC and BD (extended) intersect PQ (extended) at two points which, like X and Y, are equidistant from M.

2. Let PT and PB be two tangents to a circle, AB the diameter through B, and TH the perpendicular from T to AB. Then AP bisects TH.

3. Let the incircle (with center I) of $\triangle ABC$ touch the side BC at X, and let A' be the midpoint of this side. Then the line $A'I$ (extended) bisects AX.

2.9 Morley's theorem

One of the most surprising theorems in elementary geometry was discovered about 1904 by Frank Morley (the father of Christopher Morley, whose novel, *Thunder on the Left*, has a kink in its time sequence that appeals particularly to geometers). He mentioned it to friends in Cambridge, England, and published it twenty years later in Japan. Meanwhile it was rediscovered and presented as a problem in the Educational Times. Two solutions were sent in, one of which, by M. T. Naraniengar,† is as neat as any of the dozens that have been devised since then. The theorem states:

THEOREM 2.91. *The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle.*

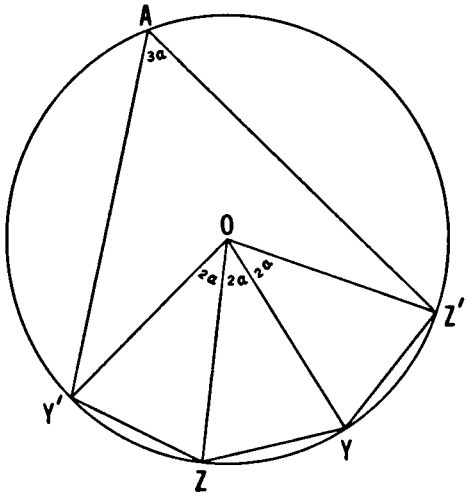


Figure 2.9A

Naraniengar's proof requires a preparatory theorem or lemma (illustrated in Figure 2.9A):

LEMMA. *If four points Y', Z, Y, Z' satisfy the conditions*

† Mathematical Questions and Their Solutions from the Educational Times (New Series), 15 (1909), p. 47.

$$Y'Z = ZY = YZ'$$

and

$$\angle YZY' = \angle Z'YZ = 180° - 2\alpha > 60°$$

then they lie on a circle. Moreover, if a point A, *on the side of the line* $Y'Z'$ *away from* Y, *is so situated that* $\angle Y'AZ' = 3\alpha$, *then this fifth point* A *also lies on the same circle.*

To prove the lemma, let the internal bisectors of the equal angles YZY' and $Z'YZ$ meet at O. Then $OY'Z$, OZY, OYZ' are three congruent isosceles triangles having base angles $90° - \alpha$. Their equal sides OY', OZ, OY, OZ' are radii of a circle with center O, and their angles at this common vertex are 2α. In other words, each of the equal chords $Y'Z$, ZY, YZ' subtends an angle 2α at the center O and consequently subtends an angle α at any point on the arc $Y'Z'$ not containing Y. This arc may be described as the locus of points (on the side of the line $Y'Z'$ away from Y) from which the chord $Y'Z'$ subtends an angle 3α. One such point is A; therefore A lies on the circle.

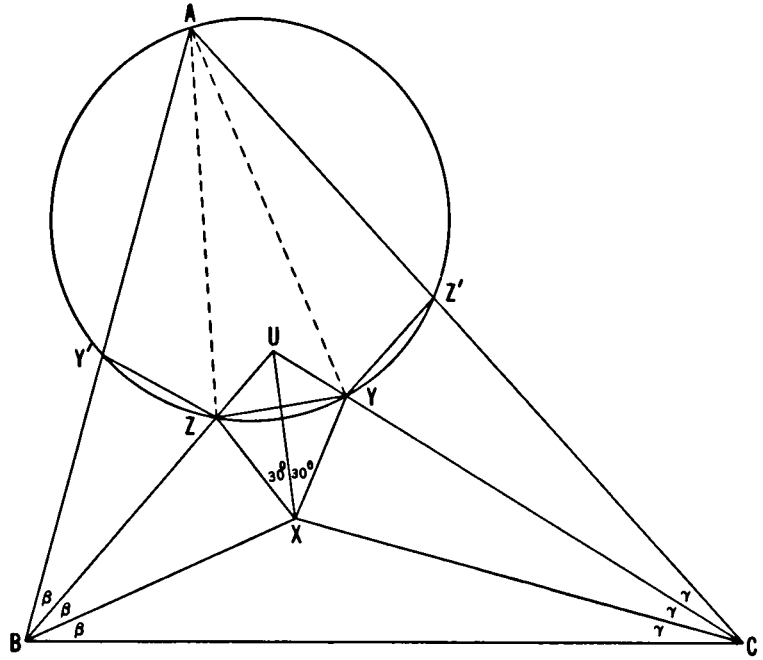


Figure 2.9B

Now we are ready to attack Theorem 2.91 itself. In Figure 2.9B, the trisectors of the angles $B = 3\beta$ and $C = 3\gamma$ meet as shown at the points U and X. In $\triangle BCU$, the angles at B and C are bisected by BX and CX; hence X is the incenter, and the angle at U is bisected

by UX. If we construct points Y and Z on the lines CU and BU so that XY and XZ make equal angles $30°$ with XU on opposite sides, then $\triangle UXY \cong \triangle UXZ$, $XY = XZ$, and since the angle at X is $60°$ it follows that $\triangle XYZ$ is equilateral.

Also $\triangle UZY$ is isosceles. Its angle at U is the same as that of $\triangle UBC$, whose other angles are 2β and 2γ; therefore the equal angles of $\triangle UYZ$ at Y and Z are each $\beta + \gamma$.

Writing $\alpha = A/3$, we deduce from $A + B + C = 180°$ that

$$\alpha + \beta + \gamma = 60°, \qquad \text{whence} \qquad \beta + \gamma = 60° - \alpha.$$

Thus

$$\angle YZU = 60° - \alpha \qquad \text{and} \qquad \angle XZU = 120° - \alpha.$$

Our next step is to mark off $BY' = BX$ on BA, and $CZ' = CX$ on CA. We now have

$$\triangle BZX \cong \triangle BZY' \qquad \text{and} \qquad \triangle CYX \cong \triangle CYZ',$$

so that

$$Y'Z = ZX = ZY = YX = YZ'.$$

Before we can apply the lemma, we still have to evaluate $\angle YZY'$ and $\angle Z'YZ$. However, this is a simple matter. Since the equal angles BZY' and BZX have equal supplements,

$$\angle UZY' = \angle XZU = 120° - \alpha$$

and

$$\angle YZY' = \angle YZU + \angle UZY' = (60° - \alpha) + (120° - \alpha)$$
$$= 180° - 2\alpha.$$

Similarly, $\angle Z'YZ = 180° - 2\alpha$; and of course $\alpha = \frac{1}{3}A < 60°$.

Applying the lemma, we deduce that the five points Y', Z, Y, Z', A all lie on a circle. Since the equal chords $Y'Z$, ZY, YZ' subtend equal angles α at A, the lines AZ and AY trisect the angle A of $\triangle ABC$. In other words, the points X, Y, Z, which were artificially constructed so as to form an equilateral triangle, are in fact the points described in Morley's theorem. The proof is now complete.

EXERCISES

1. Let the angle trisectors AZ and CX (extended) meet at V, BX and AY at W. Then the three lines UX, VY, WZ are concurrent. (That is, in the language of projective geometry, UVW and XYZ are *perspective* triangles. In general, UVW is *not* equilateral.)

2. For what kind of triangle ABC will the pentagon $AY'ZYZ'$ be regular?

3. When $\triangle ABC$ is equilateral, the four points Y', Z, Y, Z' occur among the vertices of a regular enneagon (9-gon) in which A is the vertex opposite to the side ZY.

4. For a triangle with angles 3α, 3β, 3γ and circumradius R, Morley's triangle has sides $8R \sin \alpha \sin \beta \sin \gamma$.

5. If $Z'Y = YZ = ZY'$ on the side $Z'Y'$ of a rectangle $BCZ'Y'$ whose center X forms an equilateral triangle with Y and Z, then BX and BZ trisect the right angle at B.

CHAPTER 3

Collinearity and Concurrence

> But he opened out the hinges,
> Pushed and pulled the joints and hinges,
> Till it looked all squares and oblongs
> Like a complicated figure
> In the Second Book of Euclid.
>
> *C. L. Dodgson*

After discussing some further properties of triangles and quadrangles (or quadrilaterals), we shall approach the domain of projective geometry (and even trespass a bit). A systematic development of that fascinating subject must be left for another book, but four of its most basic theorems are justifiably mentioned here because they can be proved by the methods of Euclid; in fact, three of the four are so old that no other methods were available at the time of their discovery. All these theorems deal either with *collinearity* (certain sets of points lying on a line) or *concurrence* (certain sets of lines passing through a point). The spirit of projective geometry begins to emerge as soon as we notice that, for many purposes, parallel lines behave like concurrent lines.

3.1 Quadrangles; Varignon's theorem

A *polygon* may be defined as consisting of a number of points (called *vertices*) and an equal number of line segments (called *sides*), namely a cyclically ordered set of points in a plane, with no three successive points collinear, together with the line segments joining consecutive pairs of the points. In other words, a polygon is a closed broken line lying in a

plane. A polygon having n vertices and n sides is called an n-gon (meaning literally "n-angle"). Thus we have a *pentagon* ($n = 5$), a *hexagon* ($n = 6$), and so on. In fact, the Greek name for the number n is used except when $n = 3$ or 4. In these two simple cases it is customary to use the Latin forms *triangle* and *quadrangle* rather than "trigon" and "tetragon" (although "trigon" survives in the word "trigonometry"). Obviously we should discourage the tendency to call a quadrangle a "quadrilateral". (In projective geometry, where the sides are whole lines instead of mere segments, we need both terms with distinct meanings.)

Two sides of a quadrangle are said to be *adjacent* or *opposite* according as they do or do not have a vertex in common. Similarly, two vertices are adjacent or opposite according as they do or do not belong to one side. The lines joining pairs of opposite vertices are called *diagonals*. Thus a quadrangle $ABCD$ has sides AB, BC, CD, DA, diagonals AC and BD.

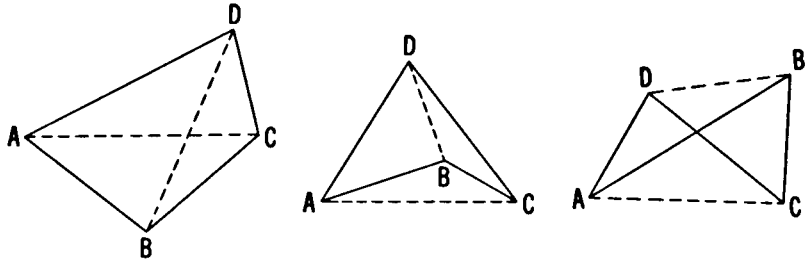


Figure 3.1A

In Figure 3.1A we see quadrangles of three obviously distinct types: a *convex* quadrangle whose diagonals are both inside, a *re-entrant* quadrangle having one diagonal inside and one outside, and a *crossed* quadrangle whose diagonals are both outside.

We naturally define the *area* of a convex quadrangle to be the sum of the areas of the two triangles into which it is decomposed by a diagonal:

$$(ABCD) = (ABC) + (CDA) = (BCD) + (DAB).$$

In order to make this formula work for a re-entrant quadrangle, we regard the area of a triangle as being *positive* or *negative* according as its vertices are named in counterclockwise or clockwise order. Thus

$$(ABC) = (BCA) = (CAB) = -(CBA).$$

For instance, the re-entrant quadrangle in the middle of Figure 3.1A has area

$$\begin{aligned}(ABCD) = (BCD) + (DAB) &= (CDA) - (CBA) \\ &= (CDA) + (ABC).\end{aligned}$$

Finally, the formula forces us to regard the area of a crossed quadrangle

as the *difference* between the areas of the two small triangles of which it is apparently composed.

When combined with the idea of directed segments (Section 2.1), the convention $(ABC) = -(CBA)$ enables us to extend our proof of Ceva's theorem and its converse (1.21 and 1.22) to cases where X or Y or Z divides the appropriate side of $\triangle ABC$ in a negative ratio, i.e., externally.

The following theorem is so simple that one is surprised to find its date of publication to be as late as 1731. It is due to Pierre Varignon (1654–1722).

THEOREM 3.11. *The figure formed when the midpoints of the sides of a quadrangle are joined in order is a parallelogram, and its area is half that of the quadrangle.*

We recall that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long as that third side. Given a quadrangle $ABCD$, let the midpoints of its sides AB, BC, CD, DA be P, Q, R, S, as in Figure 3.1B. Considering the triangles ABD and CBD, we infer that PS and QR are both parallel to the diagonal BD and equal to $\frac{1}{2}BD$. Hence the quadrangle $PQRS$ is a parallelogram;† it is often referred to as the *Varignon parallelogram* of quadrangle $ABCD$.

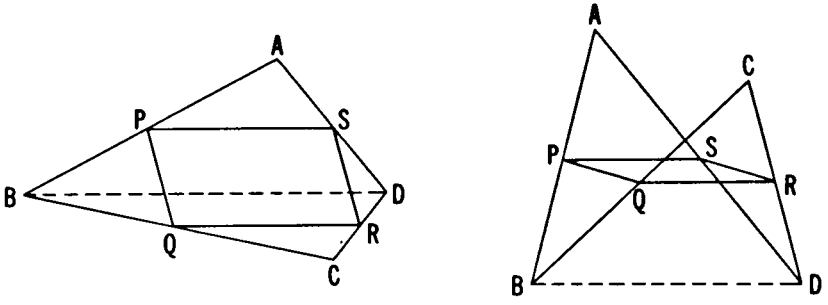


Figure 3.1B

As for the area, we have

$$\begin{aligned}(PQRS) &= (ABCD) - (PBQ) - (RDS) - (QCR) - (SAP)\\ &= (ABCD) - \tfrac{1}{4}(ABC) - \tfrac{1}{4}(CDA) - \tfrac{1}{4}(BCD) - \tfrac{1}{4}(DAB)\\ &= (ABCD) - \tfrac{1}{4}(ABCD) - \tfrac{1}{4}(ABCD)\\ &= \tfrac{1}{2}(ABCD).\end{aligned}$$

The reader may like to draw a re-entrant quadrangle $ABCD$ and verify that this decomposition is valid also in that case.

† It would still be a parallelogram if $ABCD$ were a *skew* quadrangle (not all in one plane).

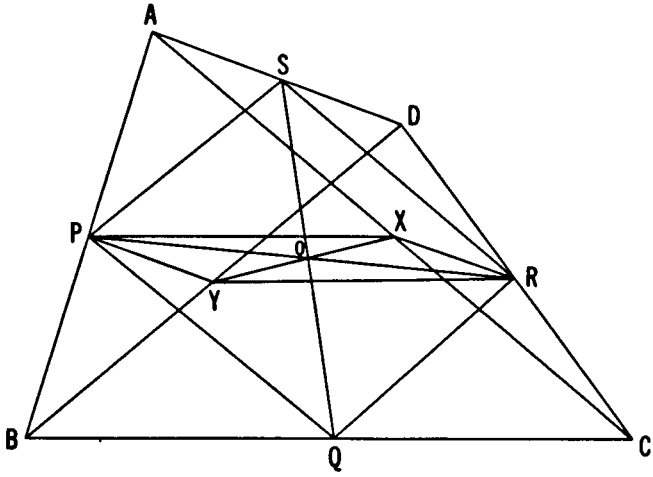


Figure 3.1C

Since the diagonals of any parallelogram bisect each other, the midpoints of PR and QS coincide at the center of the Varignon parallelogram (i.e., at that point O of Figure 3.1C). Now, just as AC and BD are the diagonals of $ABCD$, so AD and BC are the diagonals of $ABDC$. Since PR has only one midpoint, the Varignon parallelogram $PYRX$ of the new quadrangle $ABDC$ has the same center O. Hence

THEOREM 3.12. *The segments joining the midpoints of pairs of opposite sides of a quadrangle and the segment joining the midpoints of the diagonals are concurrent and bisect one another.*

(This is the first of our theorems about concurrence.)

The following result will be found useful:

THEOREM 3.13. *If one diagonal divides a quadrangle into two triangles of equal area, it bisects the other diagonal. Conversely, if one diagonal bisects the other, it bisects the area of the quadrangle.*

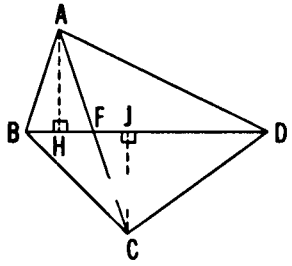


Figure 3.1D

To see why this is so, suppose BD divides $ABCD$ into two triangles DAB and BCD of equal area, as in Figure 3.1D. Since these triangles have the same "base" BD, they have equal altitudes AH and CJ. From the congruent triangles AHF and CJF, we deduce that $AF = CF$. Conversely, if $AF = CF$, then these triangles are congruent, $AH = CJ$, and $(DAB) = (BCD)$.

We are now in a position to prove the final theorem of this section:

THEOREM 3.14. *If a quadrangle $ABCD$ has its opposite sides AD and BC (extended) meeting at W, while X and Y are the midpoints of the diagonals AC and BD, then* $(WXY) = \frac{1}{4}(ABCD)$.

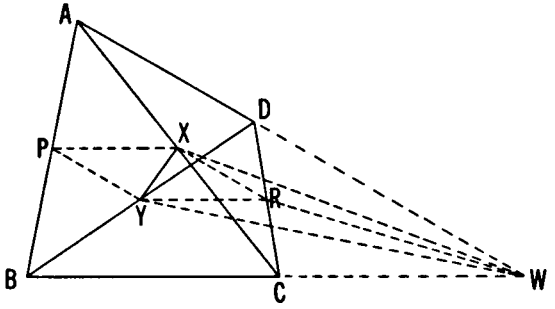


Figure 3.1E

We begin by inserting the midpoints P and R of AB and CD, as in Figure 3.1E, and drawing PX, PY, RX, RY, RW. The line RY, joining the midpoints of two sides of the triangle BCD, is parallel to BC and bisects the "other" diagonal DW of the quadrangle $DYWR$. Hence by the "converse" part of Theorem 3.13,

$$(RYW) = (YRD) = \tfrac{1}{4}(BCD).$$

In a similar manner we find that

$$(RWX) = \tfrac{1}{4}(CDA).$$

Also, by Varignon's theorem applied to the quadrangle $ABDC$,

$$\begin{aligned}(RXY) &= \tfrac{1}{2}(PYRX) = \tfrac{1}{4}(ABDC)\\ &= \tfrac{1}{4}(CAB) + \tfrac{1}{4}(BDC)\\ &= \tfrac{1}{4}(ABC) - \tfrac{1}{4}(BCD).\end{aligned}$$

Adding the last three expressions, we obtain

$$\begin{aligned}(WXY) &= (RXY) + (RYW) + (RWX)\\ &= \tfrac{1}{4}(ABC) - \tfrac{1}{4}(BCD) + \tfrac{1}{4}(BCD) + \tfrac{1}{4}(CDA)\\ &= \tfrac{1}{4}(ABC) + \tfrac{1}{4}(CDA) = \tfrac{1}{4}(ABCD).\end{aligned}$$

EXERCISES

1. The perimeter of the Varignon parallelogram equals the sum of the diagonals of the original quadrangle.

2. The sum of the squares of the sides of any quadrangle equals the sum of the squares of the diagonals plus four times the square of the segment joining the midpoints of the diagonals.

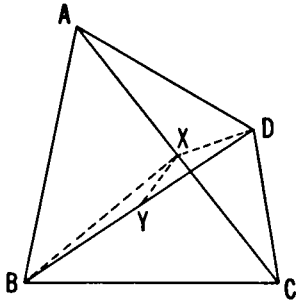


Figure 3.1F

3. For a parallelogram, the sum of the squares of the sides equals the sum of the squares of the diagonals.

4. If an isosceles trapezoid has equal sides of length a, parallel sides of lengths b and c, and diagonals of length d, then $d^2 = a^2 + bc$.

3.2 Cyclic quadrangles; Brahmagupta's formula

Let a set of E line segments, joining V points in pairs, be regarded as a "frame" in which the segments are rigid bars, pivoted at their ends but restricted to the plane. Clearly, a triangle ($E = V = 3$) is rigid, whereas a quadrangle ($E = V = 4$) has one degree of freedom: one of its angles can be increased or decreased, with a consequent change of the others. A frame is said to be "just rigid" if it is rigid but ceases to be when any one of its bars is removed. Sir Horace Lamb [**19**, pp. 93–94] gave a simple proof that a necessary (though not sufficient) condition for a frame to be just rigid is

$$E = 2V - 3.$$

For instance, $E = 5$ and $V = 4$. In this case we have a quadrangle with one diagonal; the removal of this diagonal provides the degree of freedom just mentioned.

Any four lengths a, b, c, d, each less than the sum of the other three, can be used as the sides of a convex quadrangle. The degree of freedom

enables us to increase or decrease two opposite angles till they are supplementary, and then, as we recall, the four vertices all lie on a circle. Suppose the diagonals of such a *cyclic* quadrangle are l and n (as in the first diagram of Figure 3.2A). Dissecting this quadrangle *abcd* along its diagonal l, and joining it together again after reversing the triangle *dal*, we obtain a new quadrangle *bcad*, inscribed in the same circle (as in the second diagram of Figure 3.2A). One diagonal is still l. Dissecting this cyclic quadrangle *bcad* along its other diagonal m, and joining it together again after reversing the triangle *dbm*, we obtain a third quadrangle *cabd*, inscribed in the same circle (as in the last diagram of Figure 3.2A). Since this third quadrangle could have been derived from the first by dissection along the diagonal n, its diagonals are m and n, and no further transformations of this kind are possible (except a complete reversal such as *abcd* to *dcba*). By Ptolemy's theorem, our 2.61,

$$mn = bc + ad, \qquad nl = ca + bd, \qquad lm = ab + cd.$$

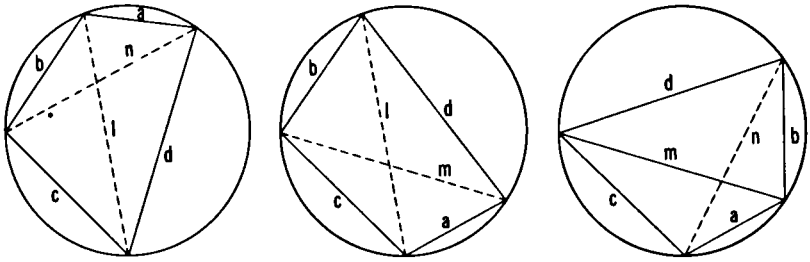


Figure 3.2A

Since these quadrangles are convex, we can regard the area of each as the sum of the positive areas of two triangles. Reversing a triangle in the manner described does not alter its positive area. Hence our three quadrangles all have the same area (though no two of them are congruent unless two of the lengths a, b, c, d happen to be equal). We summarize these remarks in the following statement:

THEOREM 3.21. *Any four unequal lengths, each less than the sum of the other three, will serve as the sides of three different cyclic quadrangles all having the same area.*

COROLLARY. *The area of a cyclic quadrangle is a symmetric function of its four sides.*

The precise nature of this symmetric function was discovered in the seventh century A.D. by the Hindu mathematician Brahmagupta:

THEOREM 3.22. *If a cyclic quadrangle has sides a, b, c, d and semi-perimeter s, its area K is given by*

$$K^2 = (s - a)(s - b)(s - c)(s - d).$$

One of the simplest methods for obtaining Brahmagupta's formula makes use of trigonometry. Consider the cyclic quadrangle $abcd$, as in Figure 3.2B, E being the vertex belonging to the sides a and b, F the vertex belonging to c and d, and n the diagonal joining the other two vertices. (We shall denote the interior angles at E and F simply by E and F.) Since $E + F = 180°$, we have

$$\cos F = -\cos E \quad \text{and} \quad \sin F = \sin E.$$

By the Law of Cosines,

$$a^2 + b^2 - 2ab \cos E = n^2 = c^2 + d^2 - 2cd \cos F,$$

whence

$$2(ab + cd) \cos E = a^2 + b^2 - c^2 - d^2. \tag{3.221}$$

Since

$$K = \tfrac{1}{2}ab \sin E + \tfrac{1}{2}cd \sin F = \tfrac{1}{2}(ab + cd) \sin E,$$

we have also

$$2(ab + cd) \sin E = 4K. \tag{3.222}$$

Squaring and adding the expressions (3.221) and (3.222), we obtain

$$4(ab + cd)^2 = (a^2 + b^2 - c^2 - d^2)^2 + 16K^2,$$

whence

$$16K^2 = (2ab + 2cd)^2 - (a^2 + b^2 - c^2 - d^2)^2.$$

By repeated application of the identity $A^2 - B^2 = (A - B)(A + B)$, we find

$$\begin{aligned}
16K^2 &= [2ab + 2cd - (a^2 + b^2 - c^2 - d^2)] \\
&\qquad \times [2ab + 2cd + (a^2 + b^2 - c^2 - d^2)] \\
&= [c^2 + 2cd + d^2 - a^2 + 2ab - b^2] \\
&\qquad \times [a^2 + 2ab + b^2 - c^2 + 2cd - d^2] \\
&= [(c + d)^2 - (a - b)^2][(a + b)^2 - (c - d)^2] \\
&= [(c + d - a + b)(c + d + a - b)] \\
&\qquad \times [(a + b - c + d)(a + b + c - d)] \\
&= (2s - 2a)(2s - 2b)(2s - 2c)(2s - 2d),
\end{aligned}$$

where $2s = a + b + c + d$. This completes the proof.

Setting $d = 0$ in Theorem 3.22, we derive Heron's formula for the area of a triangle in terms of its sides a, b, c and semiperimeter s:

$$(ABC)^2 = s(s - a)(s - b)(s - c).$$

Although this is named after Heron of Alexandria (about 60 A.D.), van der Waerden [28, pp. 228, 277] supports Bell [2, p. 58] in attributing it to Archimedes (third century B.C.).

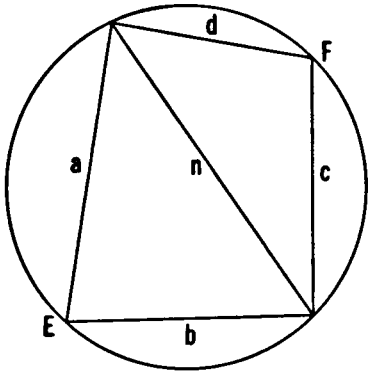


Figure 3.2B

Another discovery of Brahmagupta deals with a special kind of cyclic quadrangle:

THEOREM 3.23. *If a cyclic quadrangle has perpendicular diagonals crossing at P, the line through P perpendicular to any side bisects the opposite side.*

Referring to Figure 3.2C, where the cyclic quadrangle $ABCD$ has perpendicular diagonals AC and BD, and where the line PH, perpendicular to BC, meets DA at X, we have

$$\angle DPX = \angle BPH = \angle PCH = \angle ACB = \angle ADB = \angle XDP.$$

Hence the triangle XPD is isosceles. Similarly, so is the triangle XAP. Therefore

$$XA = XP = XD.$$

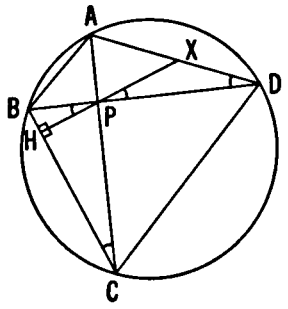


Figure 3.2C

EXERCISES

1. If a quadrangle with sides a, b, c, d is inscribed in one circle and circumscribed about another circle, its area K is given by

$$K^2 = abcd.$$

2. Find, by Heron's formula, the area of a triangle whose sides are

(i) 13, 14, 15; (ii) 3, 14, 15.

3. For a triangle ABC, express the inradius r in terms of s, $s - a$, $s - b$, $s - c$.

4. In the notation of Section 1.4,

$$r_a + r_b + r_c - r = 4R \quad \text{and} \quad (I_aI_bI_c) = 2sR.$$

5. In the notation of Figure 3.2A, $K = \dfrac{lmn}{4R}$.

6. What happens to the result of Exercise 5 when we set $d = 0$?

7. If a convex quadrangle with sides a, b, c, d is inscribed in a circle of radius R, its area K is given by

$$K^2 = \frac{(bc + ad)(ca + bd)(ab + cd)}{16R^2}.$$

8. Let two opposite sides of a cyclic quadrangle be extended to meet at V, and the other two sides to meet at W. Then the internal bisectors of the angles at V and W are perpendicular.

9. If any point P in the plane of a rectangle $ABCD$ is joined to the four vertices, we have $PA^2 - PB^2 + PC^2 - PD^2 = 0$.

10. If a quadrangle is inscribed in a circle, the product of the distances of a point on the circle from two opposite sides is equal to the product of the distances of the same point from the other two sides, and also to the product of the distances of the same point from the diagonals.

3.3 Napoleon triangles

We shall now examine some figures built with triangles and quadrangles. An easy theorem that has been surprisingly neglected is the following:

THEOREM 3.31. *Let triangles be erected externally on the sides of an arbitrary triangle so that the sum of the "remote" angles of these three triangles is 180°. Then the circumcircles of the three triangles have a common point.*

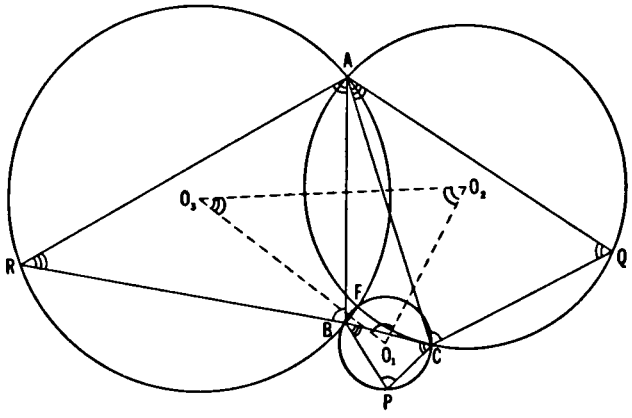


Figure 3.3A

(Here is another theorem about concurrence!) The proof is quite simple. We have, as in Figure 3.3A, triangles CBP, ACQ, BAR on the sides of the given triangle ABC, so chosen that the angles at P, Q and R satisfy the relation $P + Q + R = 180°$. Now the circumcircles of the triangles CBP and ACQ meet at C, and therefore also at another point, say F. Joining F to A, B, C, we see that

$$\angle BFC = 180° - P, \qquad \angle CFA = 180° - Q$$

and so

$$\begin{aligned} \angle AFB &= 360° - (\angle BFC + \angle CFA) \\ &= 360° - (180° - P + 180° - Q) \\ &= P + Q = 180° - R. \end{aligned}$$

Hence F lies on the circumcircle of $\triangle BAR$ as well as on the circumcircles of the other two triangles.

Two special cases are of particular interest:

THEOREM 3.32. *If the vertices A, B, C of $\triangle ABC$ lie, respectively, on sides QR, RP, PQ of $\triangle PQR$, then the three circles CBP, ACQ, BAR have a common point.*

THEOREM 3.33. *If similar triangles PCB, CQA, BAR are erected externally on the sides of* $\triangle ABC$, *then the circumcircles of these three triangles have a common point.* (Note, from the order in which we named the vertices of the similar triangles, that the angles at P, Q, R are *not* corresponding angles of these triangles.)

Theorem 3.32 has been named by Forder [13, p. 17] the *Pivot* theorem. It was discovered by A. Miquel in 1838. Changing the notation from $PQRABC$ to $ABCA_1B_1C_1$ for the sake of agreement with Figure 1.9A, we can just as easily prove it in the following slightly extended form: *If* ABC *is a triangle and* A_1, B_1, C_1 *are any three points on the lines* BC, CA, AB, *then the three circles* AB_1C_1, A_1BC_1, A_1B_1C *have a common point* P. In the special case when the circles have AP, BP, CP for diameters, $\triangle A_1B_1C_1$ is the pedal triangle of $\triangle ABC$ for the point P. Keeping ABC and P fixed, we can rigidly rotate the three lines PA_1, PB_1, PC_1 about the "pivot" P, through any angle, so as to obtain an "oblique pedal triangle" $A_1B_1C_1$. Clearly, the circles AB_1C_1, A_1BC_1, A_1B_1C continue to pass through P.

It is not necessary for A_1, B_1, C_1 to form a triangle: they may be collinear, as in Figure 2.5A. In this case A_1, B, C are three points on the lines B_1C_1, C_1A, AB_1, and the same theorem tells us that the three circles ABC, A_1B_1C, A_1BC_1 have a common point. Since the only common points of the last two circles are A_1 and P, we have now proved

THEOREM 3.34. *If four lines meet one another at six points* A, B, C, A_1, B_1, C_1, *so that the sets of collinear points are* A_1BC, AB_1C, ABC_1, $A_1B_1C_1$, *then the four circles* AB_1C_1, A_1BC_1, A_1B_1C, ABC *have a common point.*

In the special case when the first three circles have AP, BP, CP for diameters, A_1B_1 is the Simson line of P for $\triangle ABC$. Keeping ABC and P fixed, we can rigidly rotate the three lines PA_1, PB_1, PC_1 about P through any angle so as to obtain an "oblique Simson line". This line contains new "feet" A_1, B_1, C_1 such that the three lines PA_1, PB_1, PC_1 make *equal* angles (in the same sense of rotation) with the three lines BC, CA, AB.

Theorem 3.33 has an interesting corollary concerning the *triangle of centers* $O_1O_2O_3$ (Figure 3.3A). Since the sides O_2O_3, O_3O_1, O_1O_2 of this triangle are perpendicular to the common chords (or radical axes) of the pairs of circles, its angle at O_1 must be the supplement of $\angle BFC$, which means that $O_1 = P$. Similarly $O_2 = Q$ and $O_3 = R$. These are the three different angles of our three similar triangles. Hence

THEOREM 3.35. *If similar triangles PCB, CQA, BAR are erected externally on the sides of any triangle ABC, their circumcenters form a triangle similar to the three triangles.*

In particular (Figure 3.3B),

THEOREM 3.36. *If equilateral triangles are erected externally on the sides of any triangle, their centers form an equilateral triangle.*

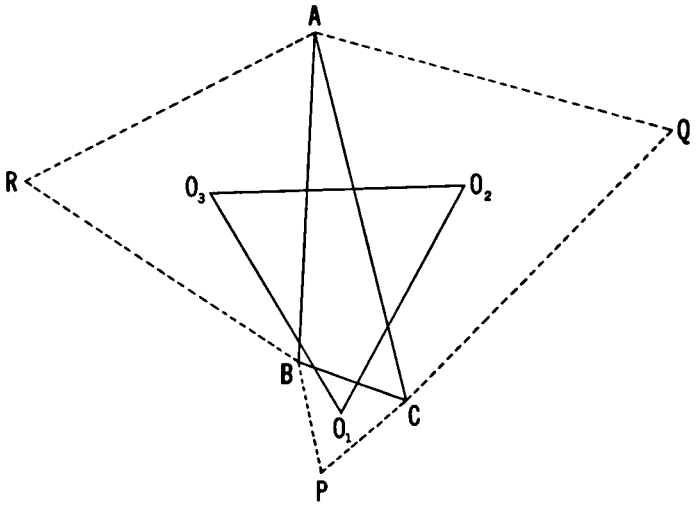


Figure 3.3B

It is known that Napoleon Bonaparte was a bit of a mathematician with a great interest in geometry. In fact, there is a story that, before he made himself ruler of France, he engaged in a discussion with the great mathematicians Lagrange and Laplace until the latter told him, severely, "The last thing we want from you, general, is a lesson in geometry." Laplace became his chief military engineer.

Theorem 3.36 has been attributed to Napoleon, though the possibility of his knowing enough geometry for this feat is as questionable as the possibility of his knowing enough English to compose the famous palindrome

ABLE WAS I ERE I SAW ELBA.

At any rate, it is convenient to name the triangle $O_1O_2O_3$ of centers (in the case when PCB, CQA, and BAR are equilateral) the *outer Napoleon triangle* of $\triangle ABC$. By analogy, if equilateral triangles are erected internally on the sides of $\triangle ABC$, as in Figure 3.3C, their centers are the vertices of the *inner Napoleon triangle* $N_1N_2N_3$. Thus Theorem 3.36 can be stated briefly as follows:

The outer Napoleon triangle is equilateral.

Yaglom [29, pp. 38, 93] proved this by another method, quite different from ours, but having the advantage that it yields also the analogous

THEOREM 3.37. *The inner Napoleon triangle is equilateral.*

A different approach which contributes also an interesting by-product applies the Law of Cosines to the triangle AO_3O_2 of Figure 3.3B. Since AO_2 is the circumradius of an equilateral triangle of side $CA = b$, its length is $b/\sqrt{3}$. Similarly, $AO_3 = c/\sqrt{3}$. Moreover,

$$\angle O_3AO_2 = A + 60°.$$

Hence

$$(O_2O_3)^2 = \tfrac{1}{3}b^2 + \tfrac{1}{3}c^2 - \tfrac{2}{3}bc \cos (A + 60°).$$

Since the vertices N_2 and N_3 of the inner Napoleon triangle can be derived from O_2 and O_3 by reflection in the lines CA and AB, respectively, and $\angle N_3AN_2 = A - 60°$, we have also

$$(N_2N_3)^2 = \tfrac{1}{3}b^2 + \tfrac{1}{3}c^2 - \tfrac{2}{3}bc \cos (A - 60°).$$

By subtraction,

$$(O_2O_3)^2 - (N_2N_3)^2 = \frac{2}{3}bc\,[\cos (A - 60°) - \cos (A + 60°)]$$

$$= \frac{4}{3}bc \sin A \sin 60° = \frac{2}{\sqrt{3}}bc \sin A$$

$$= \frac{4}{\sqrt{3}}(ABC).$$

In an analogous manner, we obtain

$$(O_1O_2)^2 - (N_1N_2)^2 = (O_3O_1)^2 - (N_3N_1)^2 = \frac{4}{\sqrt{3}}(ABC),$$

and since $O_2O_3 = O_3O_1 = O_1O_2$, we deduce

$$N_2N_3 = N_3N_1 = N_1N_2.$$

Remembering that the area of an equilateral triangle is $\sqrt{3}/4$ times the square of its side, we may formulate the "interesting by-product" as

THEOREM 3.38. *The outer and inner Napoleon triangles of any triangle ABC differ in area by* (ABC).

Actually (as we see in Figure 3.3C), the inner Napoleon triangle is

"retrograde" so that $(N_1N_2N_3)$ is negative (or zero) and the precise formula is not

$$(O_1O_2O_3) - (N_1N_2N_3) = (ABC)$$

but

$$(O_1O_2O_3) - (N_3N_2N_1) = (ABC)$$

or

$$(O_1O_2O_3) + (N_1N_2N_3) = (ABC).$$

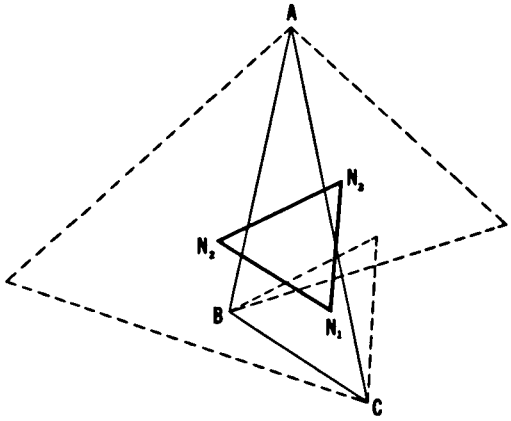


Figure 3.3C

EXERCISES

1. If squares are erected on two sides of a triangle, their circumcircles intersect on the circle whose diameter is the third side, and the centers of these three circles are the vertices of an isosceles right-angled triangle.

2. In the notation of Figure 3.3B,
   (i) The lines PO_1, QO_2, RO_3 all pass through O, the circumcenter of $\triangle ABC$;
   (ii) The lines AO_1, BO_2, CO_3 are concurrent;
   (iii) The segments AP, BQ, CR all have the same length, all pass through the common point F of the three circumcircles, and meet one another at angles of 60°. (This point is named F for Fermat, who obtained it, when no angle of $\triangle ABC$ exceeds 120°, as the point whose distances from A, B, and C have the smallest sum.)

3. In the notation of Figure 3.3C, the lines AN_1, BN_2, CN_3 are concurrent.

4. The outer and inner Napoleon triangles have the same center.

3.4 Menelaus's theorem

Menelaus of Alexandria (about 100 A.D., not to be confused with Menelaus of Sparta) wrote a treatise called *Sphaerica* in which he used a certain property of a spherical triangle; he wrote as if the analogous property of a plane triangle had been well known. Maybe it was; but since no earlier record of it has survived, we shall simply call the assertion of this property *Menelaus's theorem*. In the notation of directed segments (Section 2.1) it may be stated as follows (see Figures 3.4A, B):

THEOREM 3.41. *If points* X, Y, Z *on the sides* BC, CA, AB *(suitably extended) of* $\triangle ABC$ *are collinear, then*

$$\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = 1.$$

Conversely, if this equation holds for points X, Y, Z *on the three sides, then these three points are collinear.*

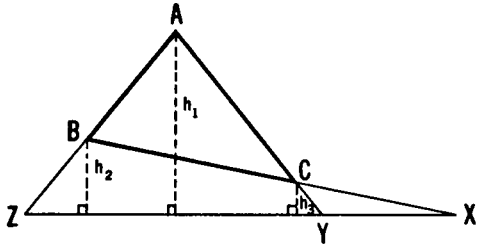


Figure 3.4A

Given the collinearity of X, Y, Z, as in Figure 3.4A or B, let h_1, h_2, h_3 be the lengths of the perpendiculars from A, B, C to the line XY, regarded as positive on one side of this line, negative on the other. From the three equations

$$\frac{BX}{CX} = \frac{h_2}{h_3}, \qquad \frac{CY}{AY} = \frac{h_3}{h_1}, \qquad \frac{AZ}{BZ} = \frac{h_1}{h_2},$$

we obtain the desired result by multiplication. (Notice that always either all three or just one of the sides of $\triangle ABC$ must be extended to accommodate the three distinct collinear points X, Y, Z.)

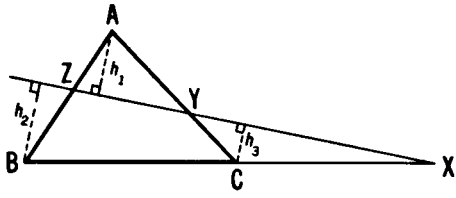


Figure 3.4B

Conversely, if X, Y, Z occur on the three sides in such a way that

$$\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = 1,$$

let the lines AB and XY meet at Z'. Then

$$\frac{BX}{CX}\frac{CY}{AY}\frac{AZ'}{BZ'} = 1.$$

Hence

$$\frac{AZ'}{BZ'} = \frac{AZ}{BZ},$$

Z' coincides with Z, and we have proved that X, Y, Z are collinear.

We observe that Menelaus's theorem provides a criterion for collinearity, just as Ceva's theorem (our 1.21 and 1.22) provides a criterion for concurrence. To emphasize the contrast, we may express Menelaus's equation in the alternative form

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = -1.$$

EXERCISES

1. The external bisectors of the three angles of a scalene triangle meet their respective opposite sides at three collinear points.

2. The internal bisectors of two angles of a scalene triangle, and the external bisector of the third angle, meet their respective opposite sides at three collinear points.

3.5 Pappus's theorem

We come now to one of the most important theorems of plane geometry. It was first proved by Pappus of Alexandria about 300 A.D., but its role in the foundations of projective geometry was not recognized until nearly sixteen centuries later. Pappus has appropriately been called the last of the great geometers of antiquity. The particular theorem that bears his name may be stated in various ways, one of which is as follows:

THEOREM 3.51. *If A, C, E are three points on one line, B, D, F on*

another, and if the three lines AB, CD, EF meet DE, FA, BC, respectively, then the three points of intersection L, M, N are collinear.

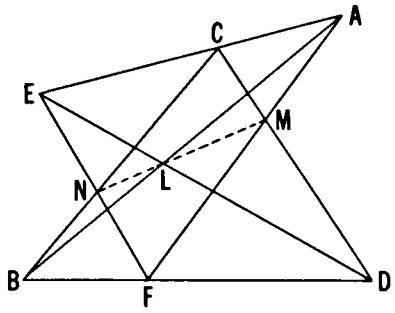


Figure 3.5A

The "projective" nature of this theorem is seen in the fact that it is a theorem of pure incidence, with no measurement of lengths or angles, and not even any reference to *order*: in each set of three collinear points it is immaterial which one lies between the other two. Figure 3.5A is one way of drawing the diagram, but Figure 3.5B is another, just as relevant. We can cyclically permute the letters A, B, C, D, E, F, provided we suitably re-name L, M, N. To avoid considering points at infinity, which would take us too far in the direction of projective geometry, let us assume that the three lines AB, CD, EF form a triangle UVW, as in Figure 3.5C. Applying Menelaus's theorem to the five triads of points

$$LDE, \quad AMF, \quad BCN, \quad ACE, \quad BDF$$

on the sides of this triangle UVW, we obtain

$$\frac{VL}{LW}\frac{WD}{DU}\frac{UE}{EV} = -1, \quad \frac{VA}{AW}\frac{WM}{MU}\frac{UF}{FV} = -1, \quad \frac{VB}{BW}\frac{WC}{CU}\frac{UN}{NV} = -1,$$

$$\frac{VA}{AW}\frac{WC}{CU}\frac{UE}{EV} = -1, \quad \frac{VB}{BW}\frac{WD}{DU}\frac{UF}{FV} = -1.$$

Dividing the product of the first three expressions by the product of the last two, and indulging in a veritable orgy of cancellation, we obtain

$$\frac{VL}{LW}\frac{WM}{MU}\frac{UN}{NV} = -1,$$

whence L, M, N are collinear, as desired. [17, p. 237.]

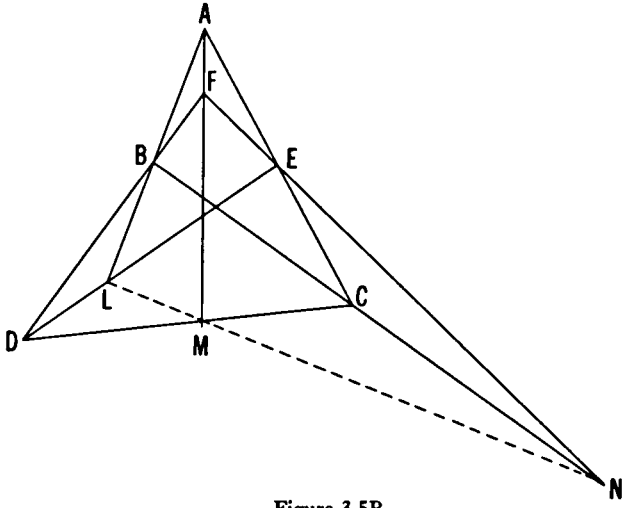


Figure 3.5B

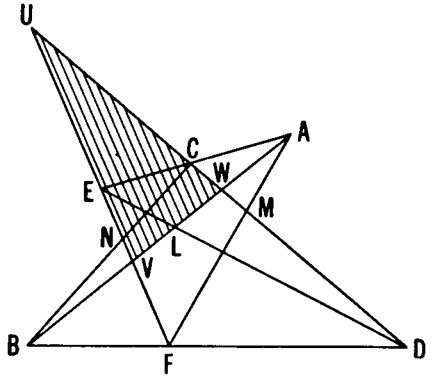


Figure 3.5C

EXERCISES

1. If A, C, E are three points on one line, B, D, F on another, and if the two lines AB and CD are parallel to DE and FA, respectively, then EF is parallel to BC.

2. If A, B, D, E, N, M are six points such that the lines AE, DM, NB are concurrent and AM, DB, NE are concurrent, what can be said about the lines AB, DE, NM?

3. Let C and F be any points on the respective sides AE and BD of a parallelogram $AEBD$. Let M and N denote the points of intersection

of CD and FA and of EF and BC. Let the line MN meet DA at P and EB at Q. Then $AP = QB$.

4. How many points and lines are named in Figure 3.5A (or Figure 3.5B)? How many of the lines pass through each point? How many of the points lie on each line?

3.6 Perspective triangles; Desargues's theorem

The geometrical theory of perspective was inaugurated by the architect Filippo Brunelleschi (1377–1446), who designed the octagonal dome of the cathedral in Florence, and also the Pitti Palace. A deeper study of the same theory was undertaken by another architect, Girard Desargues (1591–1661), whose "two-triangle" theorem was later found to be just as important as Pappus's. It can actually be deduced from Pappus's; but the details are complicated, and we can far more easily deduce it from Menelaus's.

If two specimens of a figure, composed of points and lines, can be put into correspondence in such a way that pairs of corresponding points are joined by concurrent lines, we say that the two specimens are *perspective from a point.* If the correspondence is such that pairs of corresponding lines meet at collinear points, we say that the two specimens are *perspective from a line.* In the spirit of projective geometry, Desargues's two-triangle theorem states that *if two triangles are perspective from a point, they are perspective from a line.* To avoid complications arising from the possible occurrence of parallel lines, let us be content to rephrase it as follows:

THEOREM 3.61. *If two triangles are perspective from a point, and if their pairs of corresponding sides meet, then the three points of intersection are collinear.*

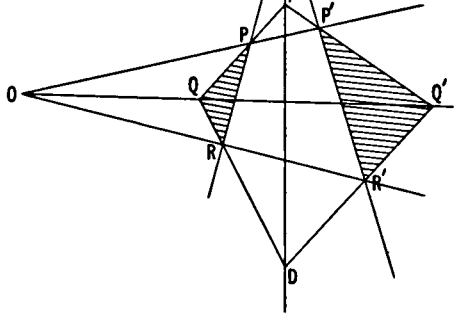


Figure 3.6A

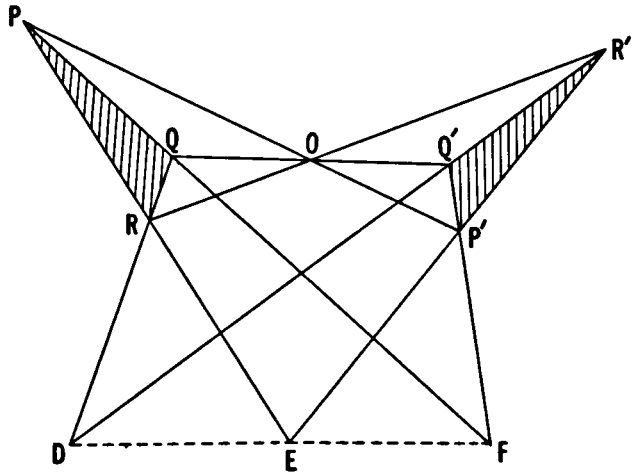


Figure 3.6B

Again we have a theorem of pure incidence. Figures 3.6A and B are two of the many ways in which the diagram can be drawn. Here $\triangle PQR$ and $\triangle P'Q'R'$ are perspective from O and their pairs of corresponding sides meet at D, E, F. (Some instances of perspective triangles have already been examined in Exercise 2 of Section 3.3, where every two of the three triangles ABC, PQR, $O_1O_2O_3$ are perspective from a point.)

For a proof, we apply Theorem 3.41 to the three triads of points

$$DR'Q', \qquad EP'R', \qquad FQ'P'$$

on the sides of the three triangles

$$OQR, \qquad ORP, \qquad OPQ,$$

obtaining

$$\frac{QD}{RD}\frac{RR'}{OR'}\frac{OQ'}{QQ'} = 1, \qquad \frac{RE}{PE}\frac{PP'}{OP'}\frac{OR'}{RR'} = 1, \qquad \frac{PF}{QF}\frac{QQ'}{OQ'}\frac{OP'}{PP'} = 1.$$

After multiplying these three expressions together and doing a modest amount of cancellation, we are left with

$$\frac{QD}{RD}\frac{RE}{PE}\frac{PF}{QF} = 1,$$

whence D, E, F are collinear, as desired. [17, p. 231.]

Desargues's theorem is easily seen to imply its converse: that *if two triangles are perspective from a line, they are perspective from a point.* Let us be content to put it thus:

THEOREM 3.62. *If two triangles are perspective from a line, and if two pairs of corresponding vertices are joined by intersecting lines, the triangles are perspective from the point of intersection of these lines.*

In declaring $\triangle PQR$ and $\triangle P'Q'R'$ to be perspective from a line, we mean that there are three collinear points†

$$D = QR \cdot Q'R', \qquad E = RP \cdot R'P', \qquad F = PQ \cdot P'Q',$$

as in Figure 3.6A. Defining $O = PP' \cdot RR'$, we wish to prove that this point O is collinear with Q and Q'. Since $\triangle FPP'$ and $\triangle DRR'$ are perspective from the point E, we can apply Theorem 3.61 to them, and conclude that the points of intersection of pairs of corresponding sides, namely

$$O = PP' \cdot RR', \qquad Q' = P'F \cdot R'D, \qquad Q = FP \cdot DR,$$

are collinear, as desired.

This is an instance of a purely "projective" proof.

EXERCISES

1. If two triangles are perspective from a point, and two pairs of corresponding sides are parallel, the two remaining sides are parallel. (In this case the two triangles are said to be *homothetic*, as in Exercise 3 of Section 1.2.)

2. How many points and lines are named in Figure 3.6A (or B)? How many of the lines pass through each point? How many of the points lie on each line?

3. Name two triangles that are perspective from (i) P, (ii) P', (iii) D.

4. What can be said about the vertices and sides of the two pentagons $DFP'OR$ and $EPQQ'R'$? Does the figure contain any other pentagons that behave in a similar manner?

5. Two non-parallel lines are drawn on a sheet of paper so that their theoretical intersection is somewhere off the paper. Through a point P, selected on the part of the paper between the lines, construct the line that would, when sufficiently extended, pass through the intersection of the given lines. What would the same construction yield if we applied it to two parallel lines?

† It will be clear from the context when a symbol such as AB denotes the whole line through points A and B and not merely the segment terminated by them. It is convenient to denote the common point of two nonparallel lines AB and DE by $AB \cdot DE$. This is less alarming than the symbol $(A \oplus B) \cap (D \oplus E)$ preferred by some authors.

3.7 Hexagons

Two vertices of a hexagon are said to be *adjacent*, *alternate*, or *opposite* according as they are separated by one side, two sides, or three sides. Thus, in a hexagon $ABCDEF$, F and B are adjacent to A, E and C are alternate to A, and D is opposite to A. The join of two opposite vertices is called a *diagonal*. Thus $ABCDEF$ has three diagonals: AD, BE, CF. Similarly the hexagon $ABCDEF$ has three pairs of *opposite sides*: AB and DE, BC and EF, CD and FA.

A given hexagon can be named $ABCDEF$ in twelve ways: Any one of its six vertices can be named A, either of the two adjacent vertices can be named B, and the rest are then determined by the alphabetical order.

Six given points, no three collinear, can be named A, B, C, D, E, F in $6! = 720$ ways. Each way determines a hexagon $ABCDEF$ having the six given points for its vertices. Hence the number of distinct hexagons determined by the six points is

$$\frac{720}{12} = 60.$$

Figure 3.7A shows three of the sixty hexagons determined by six points on a circle. Although we are accustomed to the first ("convex") kind, we must not forget or neglect the other fifty-nine possible hexagons that can be derived from the same six points.

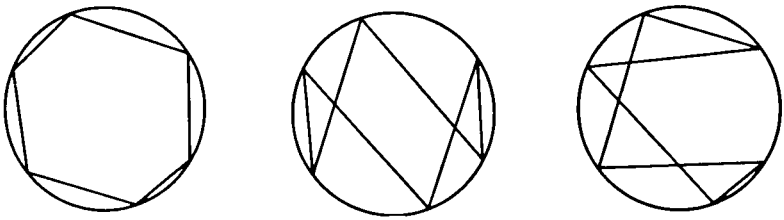


Figure 3.7A

In Section 3.1 we insisted that a polygon should have no three *successive* vertices collinear. However, other collinearities are allowed. In particular, Theorem 3.51 (Pappus's theorem) may be rephrased as follows:

If each set of three alternate vertices of a hexagon is a set of three collinear points, and the three pairs of opposite sides intersect, then the three points of intersection are collinear.

EXERCISES

1. If a hexagon $ABCDEF$ has two opposite sides BC and EF parallel to the diagonal AD, and two opposite sides CD and FA parallel to the

diagonal BE, while the remaining sides DE and AB also are parallel, then the third diagonal CF is parallel to AB, and the centroids of $\triangle ACE$ and $\triangle BDF$ coincide.

2. In how many ways can two triads of collinear points be regarded as the triads of alternate vertices of a hexagon?

3.8 Pascal's theorem

We come now to a remarkable theorem discovered by the philosopher and mathematician Blaise Pascal (1623–1662) at the age of sixteen:

THEOREM 3.81. *If all six vertices of a hexagon lie on a circle and the three pairs of opposite sides intersect, then the three points of intersection are collinear.*

Nobody knows how Pascal himself proved this, because his original proof has been lost. However, before it was lost, it was seen and praised by G. W. Leibniz (co-discoverer with Newton of the differential and integral calculus). This state of affairs challenges us to try to reconstruct the lost proof, that is, to give a proof using only the results and methods that were available in Pascal's time. One such proof, using only the first three books of Euclid, was devised by Forder [14, p. 13]; but that is a *tour de force*, and Pascal more likely used Menelaus's theorem in some such way as the following.

Figure 3.8A shows one of the many ways in which a hexagon $ABCDEF$, inscribed in a circle, may be arranged. (The reader can easily see what modifications in the argument will be needed if the arrangement is different, e.g., if the same six vertices are joined by sides in one of the fifty-nine other possible ways.) We wish to prove that the three points of intersection

$$L = AB \cdot DE, \quad M = CD \cdot FA, \quad N = BC \cdot EF$$

are collinear.

Let us assume that the three lines AB, CD, EF form a triangle UVW, as in Figure 3.8A. Applying Theorem 3.41 to the three triads of points LDE, AMF, BCN on the sides of this triangle UVW, we obtain

$$\frac{VL}{WL}\frac{WD}{UD}\frac{UE}{VE} = 1, \quad \frac{VA}{WA}\frac{WM}{UM}\frac{UF}{VF} = 1, \quad \frac{VB}{WB}\frac{WC}{UC}\frac{UN}{VN} = 1.$$

Multiplying all these expressions together, and observing that, by Theorem 2.11 (on page 28),

$$\frac{WD}{UD}\frac{UE}{VE}\frac{VA}{WA}\,\frac{UF}{VF}\frac{VB}{WB}\frac{WC}{UC} = \frac{UE \times UF}{UC \times UD}\frac{VA \times VB}{VE \times VF}\frac{WC \times WD}{WA \times WB} = 1,$$

we are left with

$$\frac{VL}{WL}\frac{WM}{UM}\frac{UN}{VN} = 1,$$

whence L, M, N are collinear, as desired.†

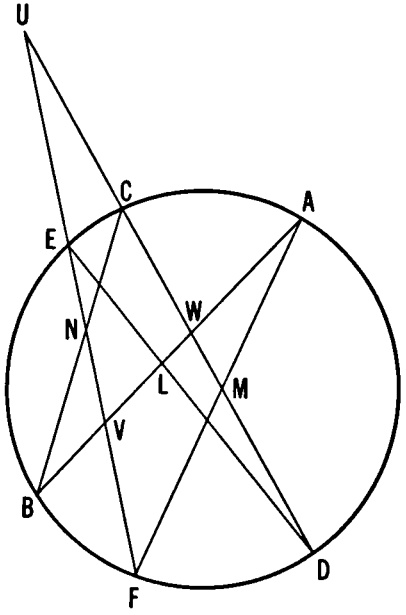


Figure 3.8A

The line containing the three points L, M, N is called the *Pascal line* of the hexagon $ABCDEF$. As we saw in Section 3.7, the same six points determine sixty hexagons; consequently they determine (in general) sixty Pascal lines. These sixty lines form a very interesting configuration: certain sets of them are concurrent, certain sets of the points of concurrence are collinear, and so on.

According to a brief *Essay pour les coniques* which has survived, Pascal was well aware that his theorem applies not only to a hexagon inscribed

† This attempt to reconstruct Pascal's proof appeared in the 18th edition of Theodor Spieker's *Lehrbuch der ebenen Geometrie* (Potsdam, 1888). See also [**17**, p. 235] or [**24**, p. 26]. For a different attempt, see Coxeter and Greitzer, *L'hexagramme de Pascal. Un essai pour reconstituer cette découverte*, Le Jeune Scientifique (Joliette, Quebec) **2** (1963), pp. 70–72.

in a *circle* but equally well to a hexagon inscribed in a *conic*. The converse theorem, proved independently by William Braikenridge and Colin MacLaurin, can be found in textbooks on Projective Geometry [e.g. **7**, p. 85]:

If the three pairs of opposite sides of a hexagon meet at three collinear points, then the six vertices lie on a conic, which may degenerate into a pair of lines (as in Theorem 3.51).

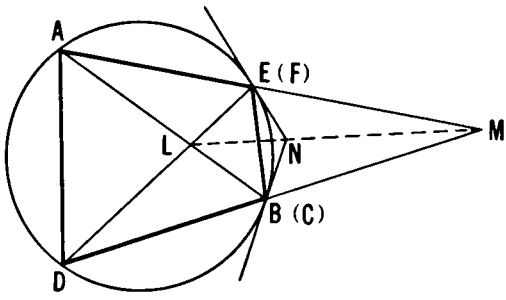


Figure 3.8B

By permitting vertices of the inscribed hexagon to coalesce and labeling them carefully, we can deduce some interesting theorems concerning inscribed pentagons and quadrangles. In such cases the side whose endpoints are made to coincide becomes a point, but the line containing it becomes the tangent to the circle (or conic) at that point. Consider, for instance, the inscribed quadrangle $ADBE$ shown in Figure 3.8B. By regarding the crossed quadrangle $ABDE$ as a degenerate hexagon with $B = C$ and $E = F$, we can apply Pascal's theorem with the conclusion that the tangents at B and E meet at N on the join of

$$L = AB \cdot DE \qquad \text{and} \qquad M = BD \cdot EA.$$

EXERCISES

1. If five of the six vertices of a hexagon lie on a circle, and the three pairs of opposite sides meet at three collinear points, then the sixth vertex lies on the same circle.

2. For a cyclic quadrangle $ABCE$ with no parallel sides, the tangents at A and C meet on the line joining $AB \cdot CE$ and $BC \cdot EA$.

3.9 Brianchon's theorem

C. J. Brianchon (1760–1854) discovered an interesting theorem (related in a subtle manner to Pascal's) involving a hexagon *circumscribed* about a conic. Brianchon's proof employs the "duality" of points and lines, which belongs to projective geometry. However, in the case when the conic is a circle, the search for a Euclidean proof became a challenging problem. This challenge was successfully answered by A. S. Smogorzhevskiĭ [27, pp. 33–34]. Before giving the details, let us prove the following lemma:

Let P' and Q' be two points on the tangents at P and Q to a circle (on the same side of the line PQ) such that $PP' = QQ'$. Then there is a circle touching the lines PP' and QQ' at P' and Q', respectively.

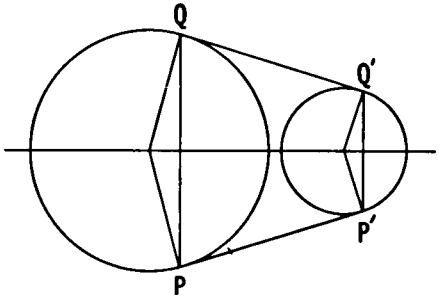


Figure 3.9A

In fact, the whole figure (Figure 3.9A) is symmetrical about the perpendicular bisector of PQ, which is also the perpendicular bisector of $P'Q'$ and a diameter of the given circle. The perpendiculars to PP' and QQ' at P' and Q' both meet this "mid-line" or "mirror" at the same point, which is the center of the desired circle.

We are now ready for Smogorzhevskiĭ's proof of

THEOREM 3.91. *If all six sides of a hexagon touch a circle, the three diagonals are concurrent (or possibly parallel).*

Let R, Q, T, S, P, U be the points of contact of the six tangents AB, BC, CD, DE, EF, FA, as in Figure 3.9B. We assume, for simplicity, that the hexagon $ABCDEF$ is "convex", so that all three diagonals AD, BE, CF are secants of the inscribed circle (and that the possibility of parallelism does not arise). On the lines EF, CB, AB, ED, CD, AF (extended), take points P', Q', R', S', T', U' so that

$$PP' = QQ' = RR' = SS' = TT' = UU'$$

(any convenient length), and construct the circles I (touching PP' and

QQ' at P' and Q'), II (touching RR' and SS' at R' and S'), III (touching TT' and UU' at T' and U'), in accordance with the lemma.

We now make use of our knowledge that two tangents to a circle from the same point have equal lengths. Since $AR = AU$ and $RR' = UU'$ we have, by addition, $AR' = AU'$. Since $DS = DT$ and $SS' = TT'$ we have, by subtraction, $DS' = DT'$. Thus both A and D are points of equal power (Section 2.2) with respect to circles II and III; and their join AD coincides with the *radical axis* of these two circles. Similarly, BE is on the radical axis of circles I and II, and CF is on the radical axis of circles III and I. As we saw in Section 2.3, *the radical axes of three non-coaxal circles, taken in pairs, are concurrent* (or possibly parallel). We have exhibited the diagonals of our hexagon as the radical axes of three circles. Since these diagonals obviously cannot coincide, the circles are non-coaxal, and the proof is complete.

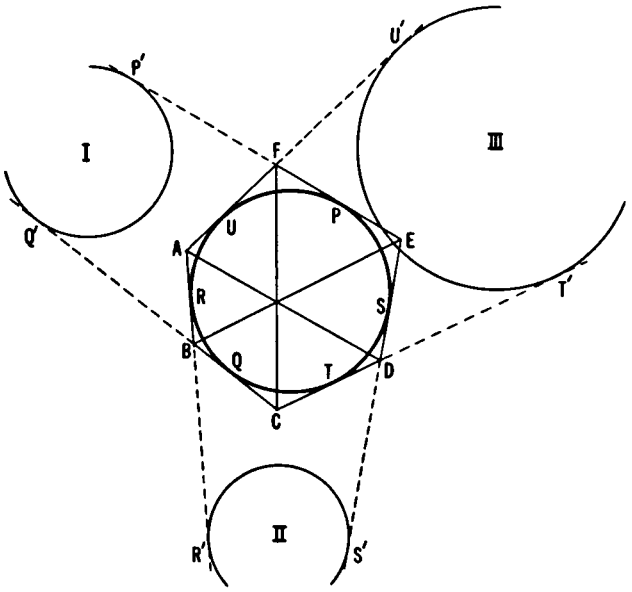


Figure 3.9B

The converse theorem, belonging to projective geometry, is as follows [7, p. 83]:

If the three diagonals of a hexagon are concurrent, the six sides touch a conic, which may degenerate into a pair of points (like the point-pair FL for the hexagon $ABDENM$ in Exercise 2 of Section 3.5).

By permitting sides of the circumscribed hexagon to coalesce and labeling them carefully, we can deduce some interesting theorems concerning circumscribed pentagons and quadrangles. In such cases the common vertex of two coincident sides becomes their point of contact with the circle (or conic).

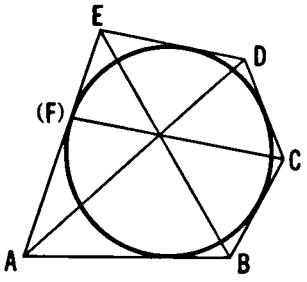


Figure 3.9C

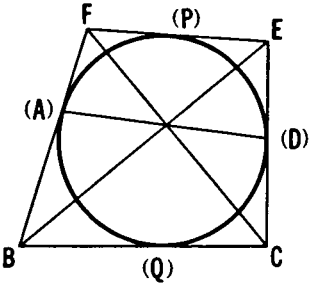


Figure 3.9D

Consider, for instance, the circumscribed pentagon $ABCDE$ shown in Figure 3.9C. By regarding it as a degenerate hexagon $ABCDEF$ with a "straight angle" at F, we can apply Brianchon's theorem with the conclusion that *the point of contact of the side EA of the circumscribed pentagon $ABCDE$ lies on the line joining C to the point of intersection $AD \cdot BE$.*

Similarly, a circumscribed quadrangle $BCEF$, whose sides FB and CE touch the circle at A and D, may be regarded as a degenerate hexagon, with the conclusion that the diagonals BE and CF of the quadrangle meet on the line AD that joins the points of contact of FB and CE with the circle.

EXERCISES

1. In Figure 3.9D, the line PQ joining the other two points of contact also passes through the intersection of the diagonals.

2. In Figure 3.9D, consider the hexagon to be $ABQCEF$. What lines are now concurrent?

3. Does Brianchon's theorem suggest a new approach to Exercise 3 of Section 1.4?

CHAPTER 4

Transformations

> By faith Enoch was translated that he should not see death; and was not found, because God had translated him: for before his translation he had this testimony, that he pleased God.
>
> *Hebrews, 11:5*

In a remark at the end of Section 1.6, we obtained the right angle between FD and OB (Figure 1.6A) by rotating the perpendicular lines HD and CB through equal angles α about D and B, respectively. In the preamble to Theorem 1.71, we observed that the two similar triangles ABC and $A'B'C'$ have the same centroid and that, since their orthocenters are H and O, $AH = 2OA'$. Finally, in the remark after Theorem 1.81, we used a half-turn to interchange the orthocenters of the two congruent triangles $A'B'C'$ and KLM. The rotation, dilatation, and half-turn are three instances of a *transformation* which (for our present purposes) means a mapping of the whole plane onto itself so that every point P has a unique image P', and every point Q' has a unique prototype Q. This idea of a "mapping" figures prominently in most branches of mathematics; for instance, when we write $y = f(x)$ we are mapping the set of values of x on the set of corresponding values of y.

Euclidean geometry is only one of many geometries, each having its own primitive concepts, axioms, and theorems. Felix Klein, in his inaugural address at Erlangen in 1872, proposed the classification of geometries according to the groups of transformations that can be applied without changing these concepts, axioms, and theorems. In particular, Euclidean geometry is characterized by the group of *similarities*; these are *angle-preserving* transformations. An important special case of a similarity is an *isometry*. This is a *length-preserving* transformation such as a rotation or, in particular, a half-turn. Isometries are at the bottom of the familiar idea of *congruence*: two figures are congruent if and only if one can be transformed into the other by an isometry.

4.1 Translation

Apart from the *identity*, which leaves all points just where they were before, the most familiar transformation is the *translation*, which preserves the distance between any two points and the direction of the line through them.

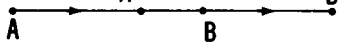


Figure 4.1A

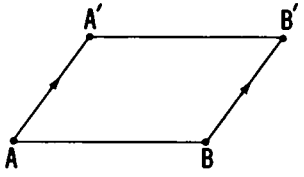


Figure 4.1B

If $A'B'$ is the translated image of a line segment AB, then either A, B, A', B' lie on a line, as in Figure 4.1A, or $AA'B'B$ is a parallelogram, as in Figure 4.1B. (In the former case we naturally speak of a *degenerate parallelogram* $AA'B'B$.) Thus the translation is determined by the directed segment AA', or equally well determined by infinitely many other segments, such as BB', having the same distance and direction. Another name for a translation is a *vector*, and we use the notation $\overrightarrow{AA'} = \overrightarrow{BB'}$. In particular, the identity may be regarded as a translation through no distance, or as the zero vector.

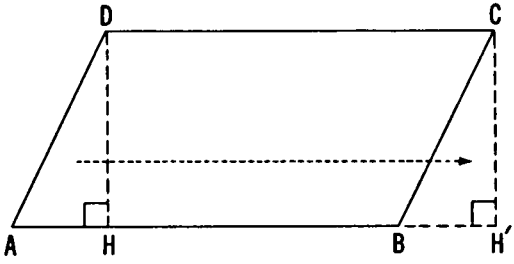


Figure 4.1C

The fact that a translation preserves the shape and size of any figure is used in the proofs of various theorems on area. For example (see Figure 4.1C), in deriving the usual formula for the area of a parallelogram $ABCD$ with an acute angle at A, we cut off a right-angled triangle AHD and stick it on again after translating it to the position $BH'C$, thus obtaining a rectangle $HH'CD$.

Figure 4.1D illustrates the problem of inscribing, in a given circle, a rectangle with two opposite sides equal and parallel to a given line segment a. This can be solved by translating the circle along either of the two equal and opposite vectors represented by a. If the old and new positions of the circle meet at B and C, these are two vertices of the desired rectangle $ABCD$, whose sides AB and DC are equal and parallel to a.

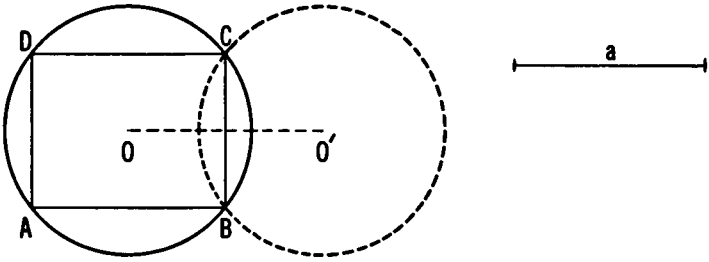


Figure 4.1D

EXERCISES

1. In $\triangle ABC$ (Figure 4.1E) "inscribe" a line segment equal and parallel to the given segment a.

2. Draw a figure to illustrate part of the infinite pattern that can be derived from a given equilateral triangle ABC by applying all the vectors consisting of an integral multiple of $\overrightarrow{AB}$ plus an integral multiple of $\overrightarrow{AC}$.

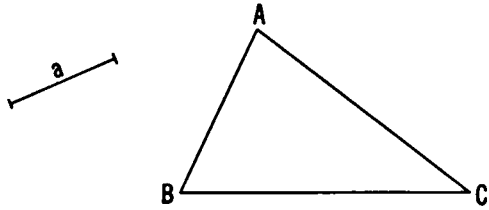


Figure 4.1E

4.2 Rotation

Another kind of transformation that preserves distance is *rotation*. Here the entire plane is turned about some point through a given angle. Thus the size and shape of any figure are kept invariant, but its points all move along arcs of concentric circles. The center (which may or may not "belong" to the figure being rotated) is the only point that remains fixed.

As an example of the use of a rotation, let us consider $\triangle ABC$ (Figure 4.2A) with equilateral triangles BPC, CQA, ARB erected (externally) on the three sides. After drawing the lines BQ and CR, which meet at F, we observe that a rotation through 60° about A takes $\triangle ARC$ into $\triangle ABQ$. Hence $\angle RFB = 60°$ and $RC = BQ$. Similar reasoning shows that $PA = CR$. Thus

$$AP = BQ = CR.$$

Moreover, since

$$\angle RFB = 60° = \angle RAB \quad \text{and} \quad \angle CFQ = 60° = \angle CAQ,$$

the quadrangles $ARBF$ and $CQAF$ are cyclic; and since $\angle BFC = 120°$ while $\angle CPB = 60°$, $BPCF$ is a third cyclic quadrangle. Therefore the circumcircles of the three triangles BPC, CQA, ARB all pass through the point F. This is called the *Fermat point* of $\triangle ABC$. Having defined it as the point of intersection of BQ and CR, we now see that it must also lie on AP.

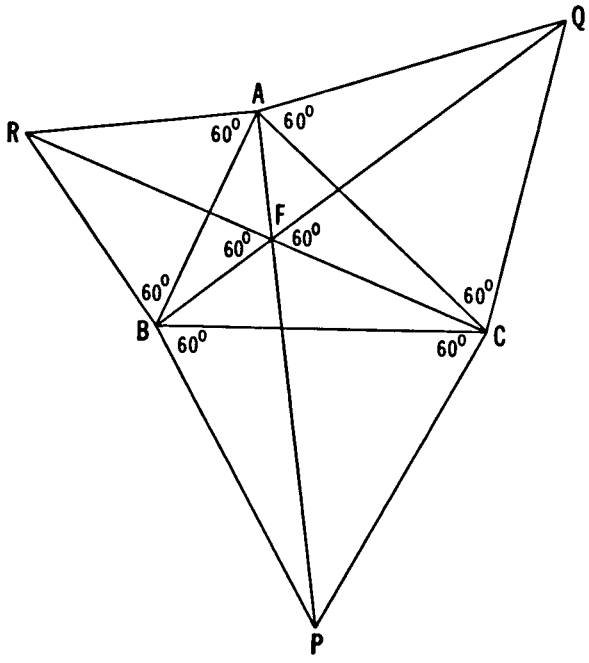


Figure 4.2A

In Euclid's proof of the theorem of Pythagoras, squares $CBIG$, $ACKJ$, $BADE$ are erected externally on the sides of the given right-angled triangle ABC and the last square is dissected into two pieces by means of the altitude CH, as in Figure 4.2B. Here O_1, O_2, O_3 are the centers of the three squares and the meaning of U, V, W, X, Y is clear. Although there are easier ways than Euclid's to prove Pythagoras's theorem itself, his figure suggests many unexpected results.

After drawing the lines AI, BJ, CD and CE, we observe that a rotation through 90° about A will take $\triangle ADC$ into $\triangle ABJ$. Therefore $BJ = DC$ and BJ is perpendicular to CD. Similarly, AI and CE are equal and perpendicular.

The similar triangles $\triangle BCX \sim \triangle BKJ$ and $\triangle CAY \sim \triangle GAI$ yield

$$\frac{CX}{b} = \frac{CX}{KJ} = \frac{BC}{BK} = \frac{a}{a+b}, \qquad \frac{CY}{a} = \frac{CY}{GI} = \frac{CA}{GA} = \frac{b}{a+b},$$

whence

$$CX = \frac{ab}{a+b} = CY.$$

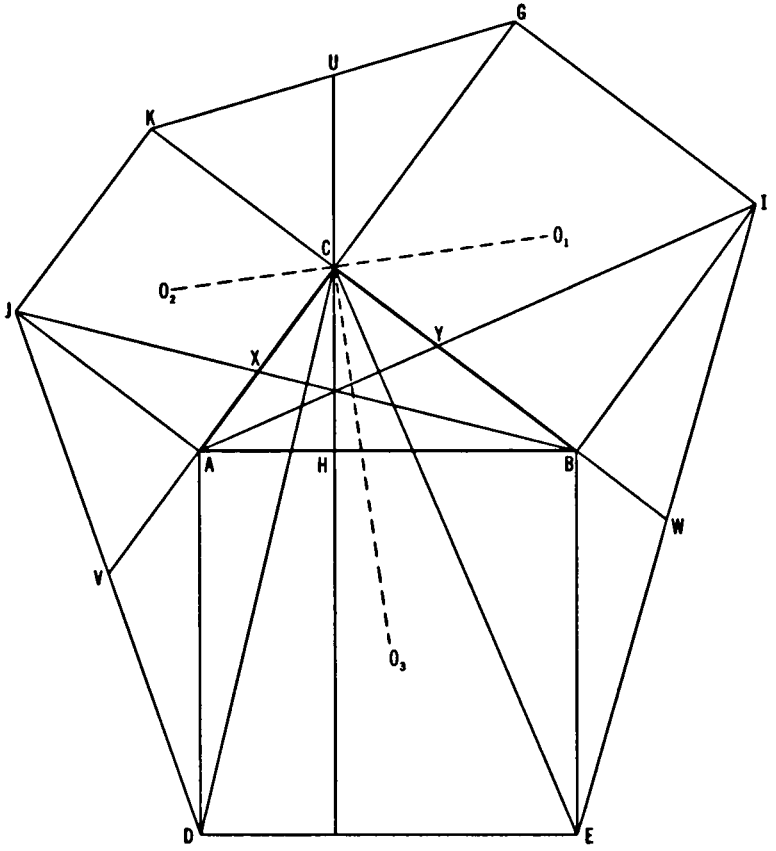


Figure 4.2B

EXERCISES

1. If squares are erected externally on the sides of a parallelogram, their centers are the vertices of a square. [**29**, pp. 96–97.]

2. In Figure 4.2B, (i) the three lines AI, BJ, CH are concurrent;

(ii) $O_1O_2 = CO_3$, and these lines are perpendicular; (iii) U, V, W are the midpoints of GK, JD, EI.

3. Construct an equilateral triangle such that a given point inside it is distant 2 units from one vertex, 3 units from a second vertex, and 4 units from the third vertex.

4.3 Half-turn

One kind of rotation shares with translations the property of transforming every line into a parallel line. This is the *half-turn* or rotation through 180°, which transforms each ray into an oppositely directed ray. Clearly, a half-turn is completely determined by its center. Since a translation transforms each ray into a parallel ray, the effect of two half-turns applied successively is the same as the effect of a translation: in brief, the "sum" of two half-turns is a translation (which reduces to the identity if the two half-turns have the same center). More precisely, if points A, B, C are evenly spaced along a line, so that B is the midpoint of AC, the half-turn about A leaves A invariant, and the half-turn about B takes A into C; thus the sum of these two half-turns is the translation $\overrightarrow{AC}$, and is the same as the sum of the half-turns about B and C.

Figure 4.3A illustrates the sum of half-turns about O_1 and O_2. The line segment AB is transformed first into $A'B'$ (oppositely directed) and then into $A''B''$; thus the sum is the translation $\overrightarrow{AA''} = \overrightarrow{BB''}$.

Many old and familiar theorems can be proved simply when half-turns are used. In Figure 4.3B, O is the common midpoint of two segments AC and BD. The half-turn about O, taking AB into CD, shows that $ABCD$ is a parallelogram. Again, in Figure 4.3C, M and N being the midpoints of AB and AC, we see that the sum of half-turns about these two points is the translation $\overrightarrow{MM''} = \overrightarrow{BC}$, whence MN is parallel to BC and half as long.

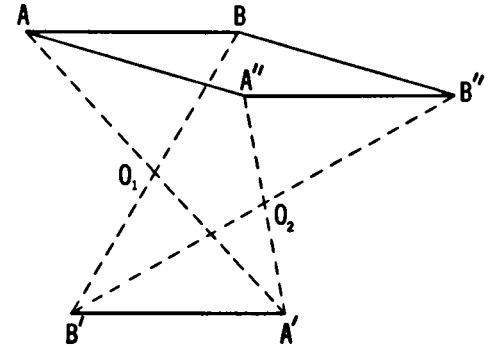


Figure 4.3A

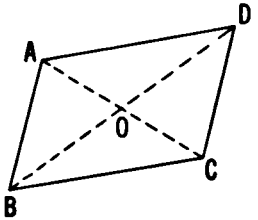


Figure 4.3B

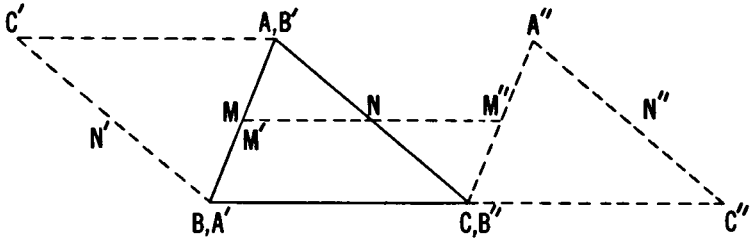


Figure 4.3C

EXERCISES

1. Let A be one of the common points of two intersecting circles. Through A, construct a line on which the two circles cut out equal chords.

2. Through a point A outside a given circle, construct a line cutting the circle at P and Q so that $AP = PQ$.

3. If the opposite sides of a hexagon are equal and parallel, the diagonals (joining opposite vertices) are concurrent.

4.4 Reflection

A third type of transformation that preserves distance is the *reflection* in a line HK, called the *mirror*. Each point on the mirror (such as H or K) is invariant, i.e. its own reflection. The reflected image of a point A not on the mirror is the point A' on the line through A perpendicular to the mirror such that AA' is bisected by the mirror. In Figure 4.4A, the segment $A'B'$ is the image of the segment AB. It is a simple matter to show that, if C is any point on the line AB, its image C' must lie on the line $A'B'$. The trapezoid $AA'B'B$ has diagonals AB' and $A'B$ which are images of each other; their common point X, being its own image, lies on the mirror HK. The properties of vertical angles permit us to label $\angle AXH = \angle B'XK$, while the congruence of $\triangle BXK$ and $\triangle B'XK$ tells us that $\angle B'XK = \angle KXB$. Hence

$$\angle AXH = \angle KXB.$$

It follows that the shortest path from an arbitrary point A to the mirror, and thence to a given point B on the same side of the mirror, is the broken line AXB. For, as we see in Figure 4.4B, if any other point Y were taken on the mirror, the path $AY + YB = A'Y + YB$ would be longer than the straight segment $A'B = AX + XB$.

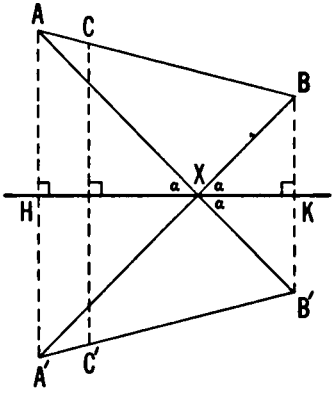


Figure 4.4A

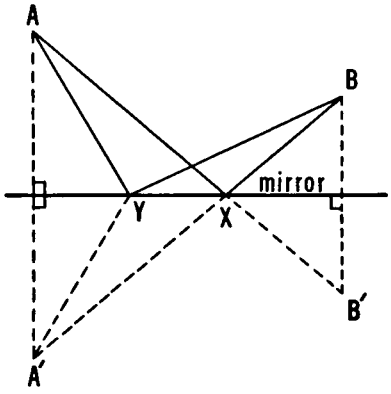


Figure 4.4B

This, incidentally, shows us how to solve geometrically a famous extremal problem without having recourse to the calculus. Physicists tell us that a ray of light travelling from a point A to a mirror and thence to another point B, will do so along a path that minimizes the travel time. In a homogeneous medium, this time is proportional to the distance travelled. So a ray of light that goes from A to B via a mirror and that meets the mirror at an angle α, leaves it making an equal angle; for, this is the result of requiring a path of minimum length. Physicists customarily measure angles from the *normal*, a line perpendicular to the mirror, instead of from the mirror itself. In Figure 4.4C, $\angle i$ is called the angle of incidence, and $\angle r$ is called the angle of reflection.

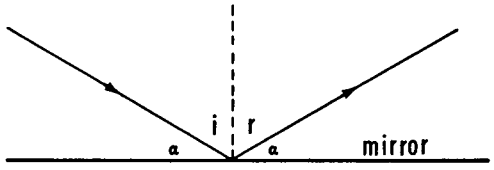


Figure 4.4C

EXERCISES

1. Given a scalene triangle ABC with sides capable of reflecting light, exactly where on the side AB should a light source be placed so that an emanating ray, after being reflected successively from the two other sides, will go back to the source? *Hint*: See Section 1.6.

2. If the base and area of a triangle are fixed, the perimeter will be minimal when the triangle is isosceles.

3. Do Exercise 1 of Section 4.3 by using a reflection.

4.5 Fagnano's problem

The properties of the mirror image can be used to derive many interesting theorems simply and in a striking fashion. We shall use these properties to solve the problem of finding the triangle of minimal perimeter inscribed in a given acute-angled triangle. This is known as Fagnano's problem†.

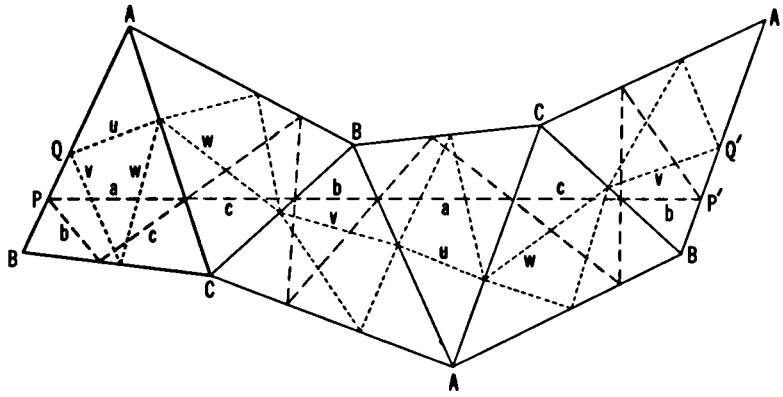


Figure 4.5A

For a solution (see Figure 4.5A), we begin with the arbitrary acute-angled triangle ABC, in which we have inscribed two triangles: the orthic triangle (dashed lines) and any other triangle (dotted lines). Let us reflect $\triangle ABC$, with contents, in its sides AC, CB, BA, AC, CB in succession. Now we inspect the diagram to see what this continued sequence of reflections has done to our triangles.

Disregarding the two points marked C, we observe a broken line $BABABA$, having angles (measured counterclockwise) $2A$ at the first point A (top left), $2B$ at the second point B (in the middle), $-2A$ at the second point A (at the bottom), and $-2B$ at the third point B (on the right). The zero sum of these four angles indicates that the final side BA is congruent *by translation* to the original side BA, and that pairs of corresponding points on these two sides will form a parallelogram such as $PP'Q'Q$.

We now recall that the altitudes of $\triangle ABC$ bisect the angles of its orthic triangle. It follows that, after the indicated reflections, the sides of the orthic triangle will, in order, lie on the straight line PP', shown in Figure 4.5A. Analogously, the sides of any other triangle, such as the

† Proposed in 1775 by Fagnano, who solved it by calculus. The proof shown here is due to H. A. Schwarz. For another proof, also using reflections, see Coxeter [6, p. 21] or Kazarinoff [18, pp. 76–77] or Courant and Robbins [4, p. 347]. Schwarz's treatment was extended from triangles to $(2n + 1)$-gons by Frank Morley and F. V. Morley, *Inversive Geometry* (Ginn, Boston, 1933), p. 37.

dotted triangle in the figure, will form a broken line reaching from Q (on the original AB) to Q' (on the final AB). Since PQ is equal and parallel to $P'Q'$, the straight segment QQ' is equal to PP', which is twice the perimeter of the orthic triangle. This is clearly shorter than the broken line from Q to Q', which is twice the perimeter of the other triangle. Hence the triangle of minimal perimeter is the orthic triangle.

4.6 The three jug problem

A curious application of reflection† is to the solution of problems requiring the division of a liquid into stated portions with what appear to be inadequate measuring devices. This application requires a preliminary account of trilinear coordinates, which we now present.

As a welcome relief from the ordinary squared paper, used for plotting points with given Cartesian coordinates, one can sometimes buy "triangulated" paper, ruled with three systems of parallel lines dividing the plane into a tessellation of small equilateral triangles. Such paper is convenient for plotting points that have given *trilinear* coordinates with respect to a (large) equilateral triangle. In the plane of such a triangle ABC, with side a and altitude h, the trilinear coordinates of a point P are defined to be the distances x, y, z of P from the three sides BC, CA, AB, regarded as positive when P is inside the triangle. We call P the point (x, y, z). Since

$$\begin{aligned} \tfrac{1}{2}ax + \tfrac{1}{2}ay + \tfrac{1}{2}az &= (PBC) + (PCA) + (PAB) \\ &= (ABC) = \tfrac{1}{2}ah, \end{aligned}$$

we have

$$x + y + z = h.$$

These coordinates are ideal for representing any situation in which three variable quantities have a constant sum. When one of the quantities stays fixed while the other two vary (with a constant sum), the point (x, y, z) moves along a line parallel to one side of the triangle. In particular, the sides themselves have the equations

$$x = 0, \quad y = 0, \quad z = 0,$$

and the vertices A, B, C have the coordinates $(h, 0, 0)$, $(0, h, 0)$, $(0, 0, h)$.

One such situation arises when h pints (or ounces) of a liquid are distributed into three vessels so that there are x pints in the first, y in the second, and z in the third. The operation of pouring liquid gradually

† M. C. K. Tweedie, *Mathematical Gazette* 23 (1939), pp. 278–282; A. I. Perel'man, Zanumatel'naya Geometria (Moscow, 1958); T. H. O'Beirne [21], pp. 49–75.

from the second vessel into the third is represented by a motion of the point (x, y, z) along a line $x =$ constant in the direction in which y decreases while z (correspondingly) increases. If each vessel can hold h pints, each coordinate can take any value from 0 to h, and we have the (trivial) problem $[h; h, h, h]$, in which the domain of operations is the whole triangular region

$$0 \leq x \leq h, \quad 0 \leq y \leq h, \quad 0 \leq z \leq h.$$

Of far greater interest is the problem $[h; a, b, c]$, where $h \geq a > b > c$. Now the three given vessels have capacities a, b, c, and the problem is to measure out a stated quantity d of liquid by repeatedly pouring from one vessel into another, either emptying the former or filling the latter (or possibly doing both things at once). The domain of operation is now restricted to the region

$$0 \leq x \leq a, \quad 0 \leq y \leq b, \quad 0 \leq z \leq c,$$

which may be a (regular or irregular) hexagon bounded by the six lines

$$x = 0, \quad x = a, \quad y = 0, \quad y = b, \quad z = 0, \quad z = c,$$

but may in special circumstances reduce to a pentagon, trapezoid, parallelogram or (as we have already seen) the whole equilateral triangle.

For instance, Figures 4.6A and 4.6B illustrate the problem $[8; 7, 6, 3]$ in which 8 pints of liquid are distributed in a given manner in vessels of capacities 7, 6, 3, and we wish to measure out (say) 4 pints. Now the domain of operation is the hexagonal region

$$0 \leq x \leq 7, \quad 0 \leq y \leq 6, \quad 0 \leq z \leq 3$$

which, being bounded by the six lines

$$x = 7, \quad z = 0, \quad y = 6, \quad x = 0, \quad z = 3, \quad y = 0,$$

has the vertices

$$(7, 1, 0), \quad (2, 6, 0), \quad (0, 6, 2), \quad (0, 5, 3), \quad (5, 0, 3), \quad (7, 0, 1)$$

or, in an abbreviated notation, 710, 260, 062, 053, 503, 701.

Figure 4.6A draws attention to the point 332 which represents a typical state: 3 pints in the first vessel, the same in the second, and 2 in the third. The broken lines radiating from this point represent the six possible operations of pouring. The path from 332 to 530 represents the act of emptying the last vessel into the first; the opposite path from 332 to 233 represents the act of filling the third vessel out of the first; and the path from 332 to 062 represents the act of emptying the first vessel into the second, which is thereby filled.

The hatched lines in Figure 4.6B show one of the various ways of passing from 332 to 440 and thus dividing the 8 pints into two equal portions. The whole path is a broken line, which proceeds always along a

direction parallel to one side of the triangle of reference and bends only when it reaches a side or vertex of the hexagon that bounds the domain of operation. Continuing this path, by the same rules, beyond 440, we would eventually reach all the points with integral coordinates on the boundary of the domain; it follows that, in the [8; 7, 6, 3] problem, *any* whole number of pints (less than 8) can be measured out.

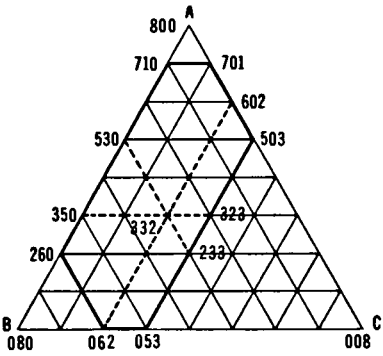


Figure 4.6A

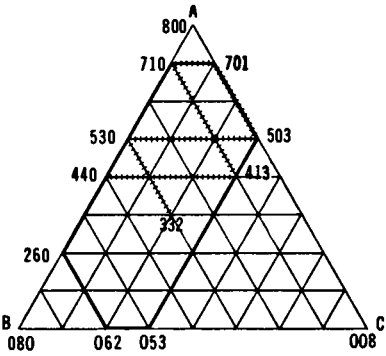


Figure 4.6B

Figure 4.6C illustrates the problem [10; 8, 7, 6], in which 10 pints of liquid have to be divided by means of vessels holding 8, 7, and 6 pints, respectively. Now we can easily measure out 1 pint, or 2 or 3 or 4. But we can never achieve 5 (unless one of the vessels is known to contain 5 initially), because the three points 055, 505, 550 form a triangular path which runs round and round like a vicious circle and cannot be entered from any other path. This kind of phenomenon arises in any problem $[h; a, b, c]$ with

$$h = 2d \geq a > b > c > d.$$

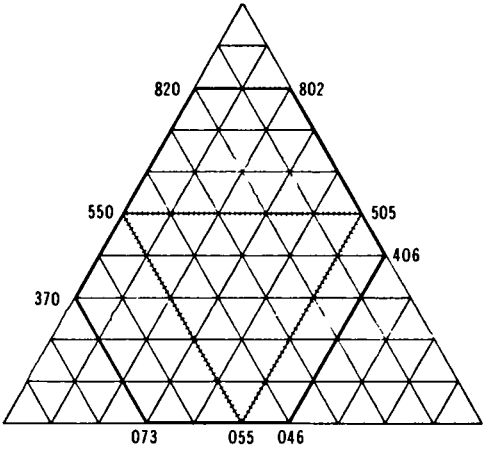


Figure 4.6C

A slightly different kind of anomaly occurs in the problem [10; 8, 6, 4] (Figure 4.6D), in which the paths that visit 550 form a pattern of small equilateral triangles and regular hexagons. This illustrates the obvious fact that an odd number of pints can never be measured with vessels whose capacities are all even. Such troubles can be expected for any problem $[h; a, b, c]$ in which the numbers a, b, c have a common divisor greater than 1.

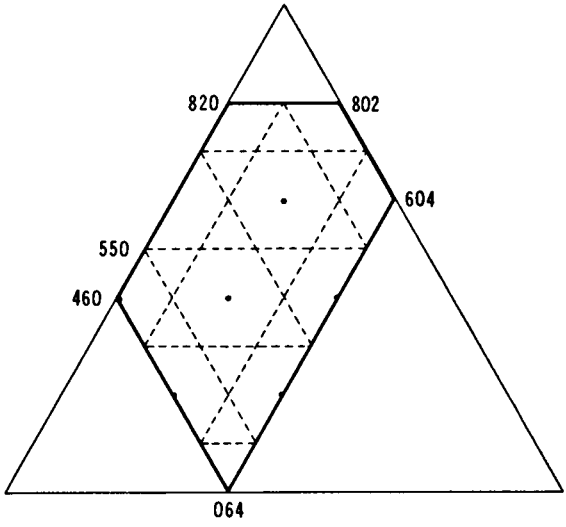


Figure 4.6D

The most famous problems $[h; a, b, c]$ are those in which

$$h = a = 2d = b + c,$$

so that the domain of operation is bounded by the parallelogram whose vertices are $a00$, $cb0$, $0bc$, $b0c$. Figures 4.6E and 4.6F show the seven-step and eight-step solutions of the problem [8; 8, 5, 3], which can be expressed as follows: Two men have a vessel filled with 8 pints of some liquid, and two empty vessels with capacities of 5 pints and 3 pints. They wish to divide the eight pints of liquid equally.

The first move must be to fill either the 5-pint vessel, as in Figure 4.6E, or the 3-pint vessel, as in Figure 4.6F. Thereafter, whenever the path reaches one of the four lines $y = 0$, $y = 5$, $z = 0$, $z = 3$, which are the sides of our parallelogram (the domain of operation), we regard that line as a mirror. In other words, we follow the path of a billiard ball which is struck so as to start out along one edge of a table having this somewhat unusual shape. (The rule of successive reflections is justified by the fact that each piece of the broken line, being parallel to a side of the triangle of reference, represents the act of pouring liquid from one

vessel into another while the third remains untouched.) We thus obtain the seven-step solution

800, 350, 323, 620, 602, 152, 143, 440

and the eight-step solution

800, 503, 530, 233, 251, 701, 710, 413, 440.

Clearly, such a problem (with $a = b + c$) can be solved whenever the integers b and c are coprime, that is, have no common divisor greater than 1.

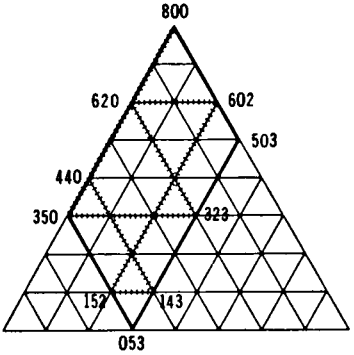


Figure 4.6E

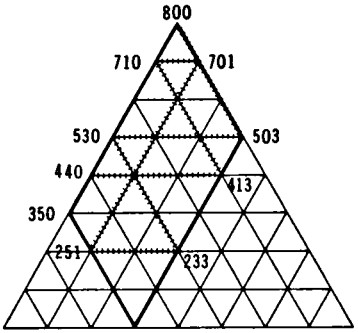


Figure 4.6F

EXERCISES

1. We are given a 12-pint vessel filled with a liquid, and two empty vessels with capacities of 9 pints and 5 pints. How can we divide the liquid into two equal portions?

2. Three men robbed a gentleman of a vase, containing 24 ounces of balsam. Whilst running away, they met a glass-seller, of whom they purchased three vessels. On reaching a place of safety, they wished to divide the booty, but found that their vessels could hold 13, 11 and 5 ounces respectively. How could they divide their booty into equal portions? [1, pp. 28, 40.]

3. Let two points P and P' have trilinear coordinates (x, y, z) and (x', y', z') with respect to a triangle ABC. If these coordinates satisfy the equations

$$xx' = yy' = zz',$$

the two points are *isogonal conjugates*:

$$\angle P'AC = \angle BAP, \quad \angle P'BA = \angle CBP, \quad \angle P'CB = \angle ACP.$$

4.7 Dilatation

The transformations presented so far have one common characteristic: they transform each figure into a congruent figure. All transformations that have this property of *preserving distance* are called congruence transformations or *isometries*.

It is possible, however, to make good use of a transformation that changes each figure into a *similar* figure. Such a *similarity* preserves angles, though it may alter distances. However, all distances are increased (or decreased) in the same ratio, called the *ratio of magnification*. Thus any line segment AB is transformed into a segment $A'B'$ whose length is given by

$$A'B' = kAB.$$

The ratio k can be greater than, equal to, or less than 1, though in the last two cases the word "magnification" is less obviously appropriate. Similarities include, as special cases, isometries, for which $k = 1$.

These remarks can be made more precise by defining a similarity to be *a transformation that preserves ratios of distances*. For this implies that it preserves both collinearity and angles.

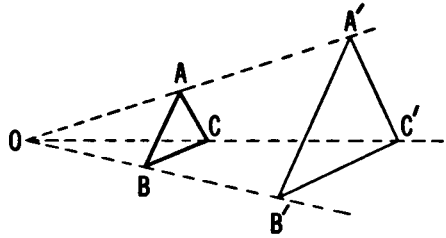


Figure 4.7A

The simplest kind of similarity is a *dilatation*, which *transforms each line into a parallel line*. Any dilatation that is not merely a translation is called a *central* dilatation, because all the lines joining corresponding points of the figure and its image are concurrent. To see why this is so, examine Figures 4.7A and B, in which the corresponding segments AB and $A'B'$ (lying on parallel lines) satisfy the vector equation

$$\overrightarrow{A'B'} = \pm\overrightarrow{AB}.$$

For any point C that forms a triangle with A and B, the image C' is where the line through A' parallel to AC meets the line through B' parallel to BC. If the dilatation is not a translation, the lines AA' and BB' are not parallel but meet at a point O, such that either

$$\overrightarrow{OA'} = k\overrightarrow{OA} \quad \text{and} \quad \overrightarrow{OB'} = k\overrightarrow{OB},$$

as in Figure 4.7A, or

$$\overrightarrow{OA'} = -k\overrightarrow{OA} \quad \text{and} \quad \overrightarrow{OB'} = -k\overrightarrow{OB},$$

as in Figure 4.7B. Remembering that parallel lines cut transversals into proportional segments, we easily deduce that C' lies on OC; in fact,

$$\overrightarrow{OC'} = \pm k\overrightarrow{OC}.$$

Varying Figure 4.7A by making O recede far away to the left, we see how a *translation* arises as the limiting form of a central dilatation $\overrightarrow{A'B'} = k\overrightarrow{AB}$ when k tends to 1. Still more easily, we can change Figure 4.7B so as to make O the midpoint of AA'; thus the central dilatation $\overrightarrow{A'B'} = -k\overrightarrow{AB}$ includes, as a special case, the *half-turn*

$$\overrightarrow{A'B'} = -\overrightarrow{AB},$$

for which $ABA'B'$ is a parallelogram with center O.

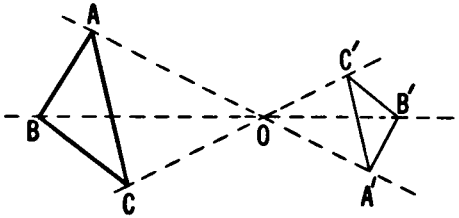


Figure 4.7B

EXERCISES

1. What is the locus of the midpoint of a segment of varying length such that one end remains fixed while the other end runs around a circle?

2. Given an acute-angled triangle ABC, construct a square with one side lying on BC while the other two vertices lie on CA and AB, respectively.

4.8 Spiral similarity

If a figure is first dilated and then translated, the final figure and the original figure still have corresponding lines parallel, so that the result is simply a dilatation. More generally, and for the same reason, the sum of any two dilatations (i.e. the effect of first performing one, then the other dilatation) is a dilatation. On the other hand, if a figure is first dilated and then rotated, corresponding lines are no longer parallel. Thus the sum of a dilatation and a rotation (other than the identity or a half-turn) is not a dilatation, though it is still a *direct similarity*, preserving angles in both magnitude and sign.

The sum of a central dilatation and a rotation *about the same center* is called a "dilative rotation" or *spiral similarity*. This little known transformation can be used in the solution of many problems.

If, as in Figure 4.8A, a spiral similarity with center O takes AB to $A'B'$, then $\triangle OAB$ and $\triangle OA'B'$ are directly similar, and

$$\angle AOA' = \angle BOB'.$$

Moreover, as in the case of a simple dilatation, the ratio of magnification is

$$k = \frac{OA'}{OA} = \frac{A'B'}{AB}.$$

Since any spiral similarity is completely determined by its center O, ratio k, and angle of rotation θ, let us agree to denote it by the symbol

$$O(k, \theta).$$

(As usual, a rotation in a counterclockwise direction will be positive, a rotation in a clockwise direction negative.) In particular, $O(k, 0°)$ and $O(k, 180°)$ are dilatations of the types illustrated in Figures 4.7A and 4.7B, respectively, and $O(1, \theta)$ is a rotation.

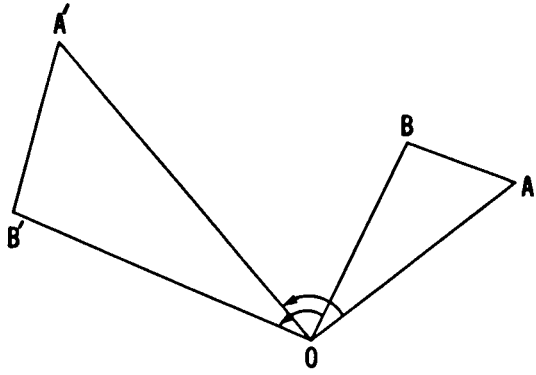


Figure 4.8A

As an example of the use of spiral similarities, let us prove

THEOREM 4.81. *If squares, with centers* O_1, O_2, O_3, *are erected externally on the sides* BC, CA, AB *of* $\triangle ABC$, *then the line segments* O_1O_2 *and* CO_3 *are equal and perpendicular.*

In the notation of Figure 4.8B, the spiral similarity $A(\sqrt{2}, 45°)$ will transform $\triangle CAO_3$ into $\triangle KAB$, and the spiral similarity $C(\sqrt{2}, -45°)$ will transform $\triangle O_1CO_2$ into $\triangle BCK$. Since the transforms have the side BK in common, arising from O_3C and O_1O_2, respectively, and since the magnification ratio is the same in both transformations, these

two sides of the original triangles must have been equal to start with. Also, since the angle between the transforms of O_2C and O_1O_3, by similarities involving rotations through 45° and −45°, is zero, these lines must originally have been perpendicular. The proof is now complete. (Notice that the three lines AO_1, BO_2, CO_3, being the altitudes of $\triangle O_1O_2O_3$, are concurrent.)

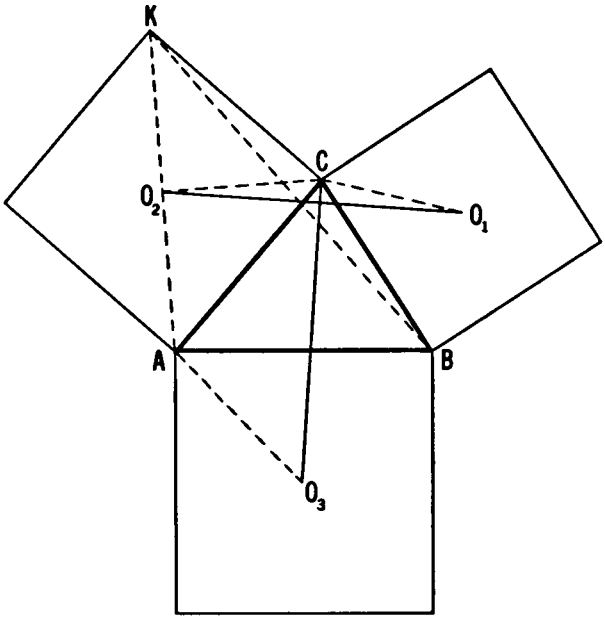


Figure 4.8B

Having defined a spiral similarity as the sum of a central dilatation and a rotation about the same center, we naturally wonder what is the sum of a central dilatation and a rotation whose centers are distinct. the simple and surprising answer—*a spiral similarity*—is a consequence of the fact that no more complicated kind of direct similarity exists:

THEOREM 4.82. *Any two directly similar figures are related either by a translation or by a spiral similarity.*

To prove this, consider two corresponding segments AB and $A'B'$ of directly similar figures. If AB is parallel to $A'B'$ and of the same length, then the transformation is a translation. To see this let C be any point not on AB and let C' be its image. Then, from the direct similarity of the figures, we may conclude that triangles ABC and $A'B'C'$ are congruent; their corresponding sides are parallel. It follows that all segments joining points and their images are parallel and equal, so the transformation is a translation.

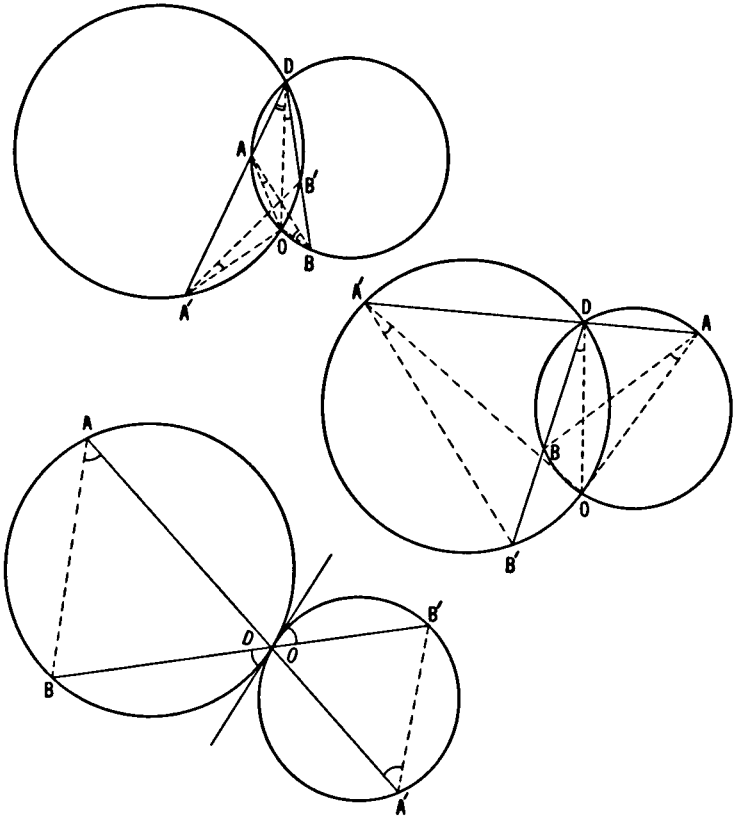


Figure 4.8C

Next, suppose AB and $A'B'$ are not of the same length. (If the four points A, B, A', B' do not form a quadrangle, pick a new pair of corresponding segments so that they do, and name these AB and $A'B'$. For instance, if B lies on AA', as in Figure 4.8D, use the midpoint of AB instead of A, and the midpoint of $A'B'$ instead of A'.) Then the lines AA' and BB' meet at a point D, as in Figure 4.8C. Let the circles ABD and $A'B'D$, which have the common point D, meet again at O (or, if they have D as a point of contact, let O be another name for D). By comparing the angles OAB, ODB, ODB', and $OA'B'$, we see that $\angle OAB = \angle OA'B'$. Similarly, $\angle OBA = \angle OB'A'$. Thus $\triangle OAB$ and $\triangle OA'B'$, being directly similar, are related by the spiral similarity $O(k, \theta)$ where

$$k = \frac{OA'}{OA} \quad \text{and} \quad \theta = \angle AOA'.$$

In other words, every direct similarity that is not a translation has an

invariant point. Moreover, *the invariant point is unique.* For, two such points, say A and B, would yield an invariant segment AB. Since

$$k = \frac{AB}{AB} = 1,$$

the similarity would be an isometry leaving two points fixed. If this transforms a triangle ABC into ABC', we can locate C' as lying on the circles with centers A and B, radii AC and BC. Thus the only isometries leaving A and B invariant are the identity, which is a translation (through distance zero), and a reflection, which is not direct (because it reverses the sign of an angle).

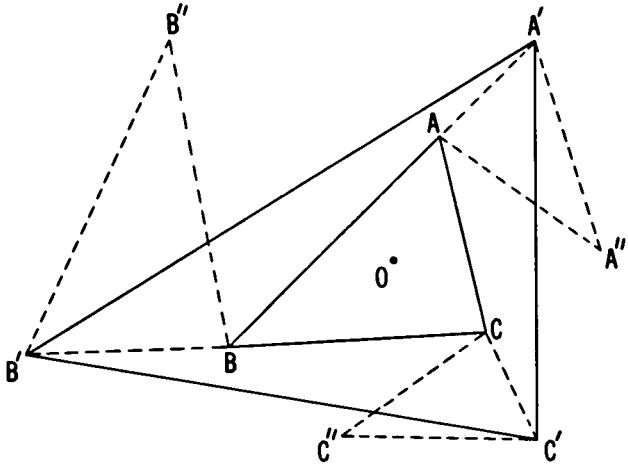


Figure 4 8D

For instance, if two maps of the same state, on different scales, are drawn on tracing paper and superposed,† there is just one place that is represented by the same spot on both maps.

These ideas have been developed by Julius Petersen (1880) and P. H. Schoute (1890)‡ into a very beautiful theorem, of which the following is a special case:

THEOREM 4.83. *If ABC and $A'B'C'$ are two directly similar triangles, while $AA'A''$, $BB'B''$, $CC'C''$ are three directly similar triangles, then $\triangle A''B''C''$ is directly similar to $\triangle ABC$.*

† Here the word "superposed" should be interpreted to mean that the smaller scale map lies entirely within the larger scale map. In this case it is easy to show that the center of the spiral similarity is indeed a point within the state.

‡ See J. Petersen [25, p. 74] or H. G. Forder [12, p. 53].

If $\triangle ABC$ and $\triangle A'B'C'$ are congruent by translation, this is obvious. if not, let $O(k, \theta)$ be the unique spiral similarity that transforms ABC into $A'B'C'$, so that

$$k = \frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC},$$

$$\theta = \angle AOA' = \angle BOB' = \angle COC',$$

as in Figure 4.8D. It follows that

$$\triangle OAA' \sim \triangle OBB' \sim \triangle OCC'.$$

But we are assuming

$$\triangle AA'A'' \sim \triangle BB'B'' \sim \triangle CC'C''.$$

Hence

$$\triangle OAA'' \sim \triangle OBB'' \sim \triangle OCC'';$$

$$\frac{OA''}{OA} = \frac{OB''}{OB} = \frac{OC''}{OC} = k',$$

$$\angle AOA'' = \angle BOB'' = \angle COC'' = \theta'$$

and there is a spiral similarity $O(k', \theta')$ relating $\triangle ABC$ to $\triangle A''B''C''$.

Another special case of the Petersen–Schoute theorem, proved in the same way, is

THEOREM 4.84. *When all the points P on AB are related by a similarity to all the points P' on $A'B'$, the points dividing the segments PP' in a given ratio are distinct and collinear or else they all coincide.*

EXERCISES

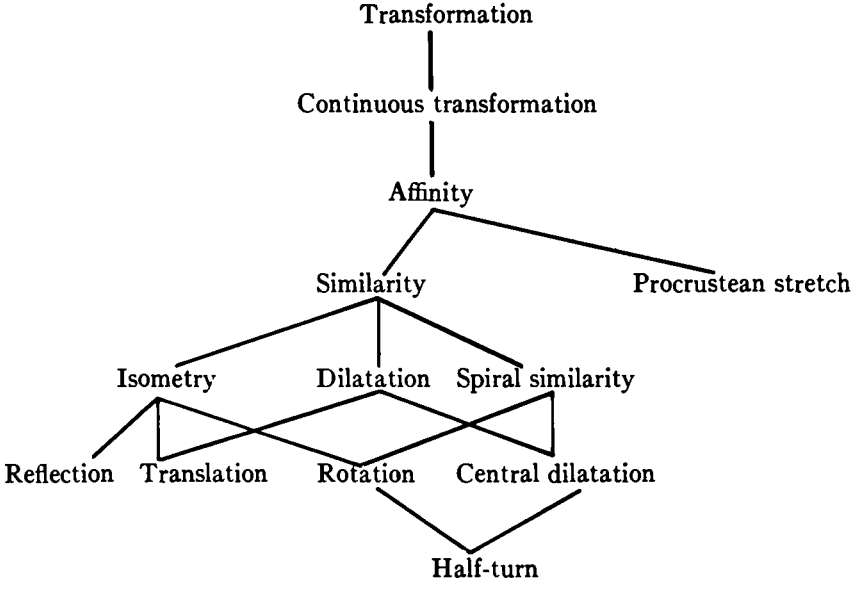
1. If $\triangle ABC$ is subjected to a spiral similarity about its vertex A in such a way that the vertex B travels along the line BC, then the vertex C travels along a line.

2. If $\triangle ABC$ is scalene, its inner Napoleon triangle $N_1N_2N_3$ is retrograde; i.e., its sense of orientation is opposite to that of $\triangle ABC$ and $\triangle O_1O_2O_3$. (This was stated without proof in Section 3.3.)

4.9 A genealogy of transformations

It is significant that all the transformations which we have been discussing are one-to-one correspondences of the whole set of points in

the plane with itself. Among these we have considered only *continuous transformations* (or "homeomorphisms"), that is, transformations which map nearby points into nearby points.† Among the continuous transformations (which are, in a sense, the subject of O. Ore's book [22]) we have discussed the *affinities* (or "affine transformations"), which preserve collinearity and thus take parallel lines into parallel lines. Among the affinities we have considered only *similarities*, which preserve ratios of distances, but we have not touched upon the more bizarre varieties such as the "Lorentz transformation" or *Procrustean stretch* (which changes a circle into an ellipse of the same area). The particular similarities that we have considered are *isometries*, which preserve distance, *dilatations*, which transform each line into a parallel line, and *spiral similarities* which (like some isometries and some dilatations) leave one point fixed and preserve the sense of rotation (counterclockwise or clockwise). These categories overlap somewhat: among the isometries, we have considered reflections, translations (which are dilatations according to the above definition), and rotations (which are spiral similarities with ratio of magnification 1). The remaining dilatations are central dilatations (which are spiral similarities involving the zero rotation). Finally, half-turns are both rotations (through 180°) and central dilatations. All these relationships can be neatly summarized in a genealogical tree, where each "child" is a specialization of its "parent".



† More precisely: If A is a point and A' its image under a continuous transformation, then the image B' of B will fall into an arbitrarily small circle about A' provided only that B is sufficiently close to A.

EXERCISES

In terms of Cartesian coordinates, a Procrustean stretch transforms each point (x, y) into (x', y'), where $x' = kx$, $y' = k^{-1}y$. Write down analogous expressions for:

1. The translation that takes $(0, 0)$ to (a, b).
2. Reflection in the y-axis.
3. Reflection in the line $x - y = 0$.
4. The half-turn about the origin O.
5. The central dilatation $O(k, 0°)$.
6. The spiral similarity $O(k, 90°)$.
7. An isometry that has not yet been mentioned.
8. A similarity that has not yet been mentioned.
9. A continuous transformation that is not an affinity.
10. A transformation that is not continuous.

CHAPTER 5

An Introduction to Inversive Geometry

> We place a spherical cage in the desert, enter it, and lock it. We perform an inversion with respect to the cage. The lion is then in the interior of the cage, and we are outside.
>
> *H. Petard*†

In this chapter we relax (to the smallest possible extent) our restriction to transformations that are one-to-one over the whole Euclidean plane: we allow just one point O to have no transform. More precisely, we consider a fixed circle with center O, and "invert" in this circle. What happens is that circles through O are transformed into lines, and other circles into circles. (Problems concerning circles are often simplified by thus changing some of the circles into lines. More complicated figures suffer radical changes in shape.)

5.1 Separation

The following theorem was considered sufficiently challenging to be used as a question in the William Lowell Putnam Competition for 1965. Our treatment is a distillation of the various solutions submitted.

† For other types of big game hunting, see *Am. Math. Monthly*, Aug.–Sept. 1938, pp. 446–447.

THEOREM 5.11. *If four points A, B, C, D do not all lie on one circle or line, there exist two non-intersecting circles, one through A and C, the other through B and D.*

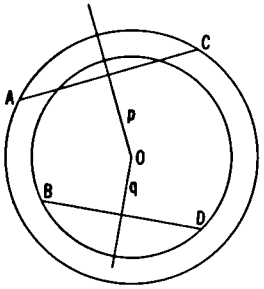


Figure 5.1A

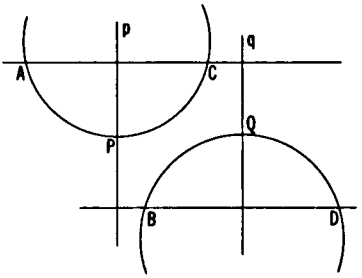


Figure 5.1B

To prove this, notice first that p, the perpendicular bisector of segment AC, cannot coincide with q, the perpendicular bisector of segment BD. If the lines p and q intersect, as in Figure 5.1A, their common point O is the center of two *concentric* circles, one through A and C, the other through B and D. If, instead, p and q are parallel, as in Figure 5.1B, so also are the lines AC and BD. Consider points P and Q, on p and q respectively, midway between the parallel lines AC and BD. Clearly, the circles APC and BQD have no common point.

Two distinct point pairs, AC and BD, are said to *separate* each other if A, B, C, D lie on a circle (or on a line) in such an order that either of the arcs AC (or the line segment AC) contains one but not both of the remaining points B and D. The customary symbol for this relation is

$$AC \,//\, BD$$

which can be written equally well in seven other ways, such as $AC \,//\, DB$ or $BD \,//\, AC$.

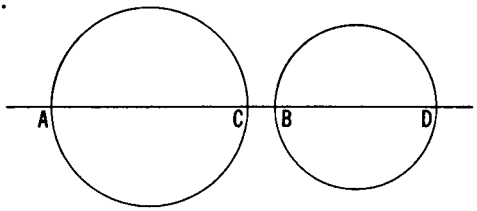


Figure 5.1C

If two point pairs, AC and BD, on a line or on a circle, do *not* separate each other, it is easy to draw two non-intersecting circles, one through A and C, the other through B and D. In the case of collinear points (Figure 5.1C), we can use circles having the line segments AC and BD as diameters. In the case of concyclic points with $AB \,//\, CD$ and $BC < AD$ (Figure 5.1D), we can take the centers to be the points of intersection of the line BC with the perpendicular bisectors of AC

and BD, respectively. A slightly different treatment is needed in the easy case when $ABCD$ is a rectangle.

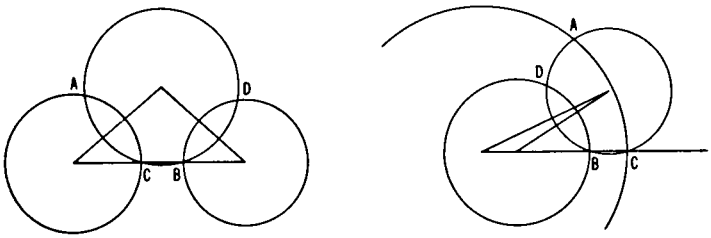


Figure 5.1D

If, on the other hand, $AC \,//\, BD$, any circle through A and C, but not through B, "separates" B and D, in the sense that one of those two points is inside and the other outside. Therefore the given circle through A and C intersects every circle through B and D.

The contrapositive form of Theorem 5.11 tells us that, if every circle through two given points has at least two points in common with every circle through two other given points, the four given points must be collinear (Figure 5.1E) or concyclic (Figure 5.1F). Under such circumstances, as we have seen, the two pairs of points separate each other. These remarks enable us to redefine separation in a manner that is symmetrical and does not presuppose our knowledge of whether the four points are collinear or concyclic or neither:

Two distinct point pairs, AC and BD, are said to *separate* each other if *every circle through A and C intersects (or coincides with) every circle through B and D.*

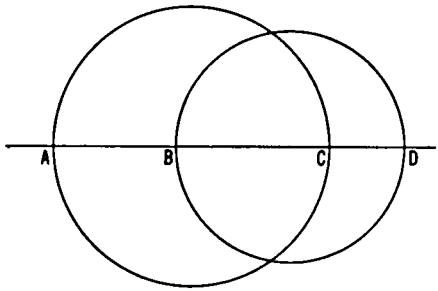


Figure 5.1E

There is actually a third way to characterize separation, without mentioning circles at all:

THEOREM 5.12. *The mutual distances of four distinct points A, B, C, D satisfy*

$$AB \times CD + BC \times AD \geq AC \times BD,$$

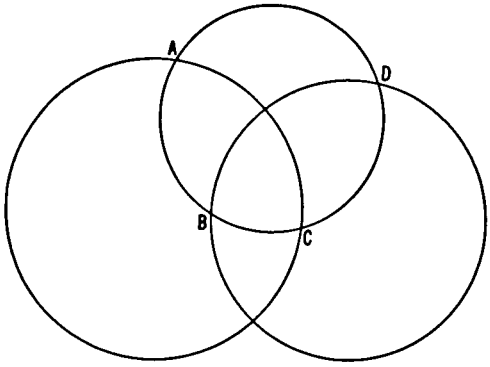


Figure 5.1F

The proof has to be followed with some care, but is interesting. Let us first dispose of the case when the four points all lie on one line, so that we can temporarily use the notation of directed segments (positive or negative, as in Section 2.1). Writing

$$AD = x, \quad BD = y, \quad CD = z,$$

so that

$$AB = x - y, \quad BC = y - z, \quad AC = x - z,$$

we have

$$AB \times CD + BC \times AD = (x-y)z + (y-z)x = (x-z)y$$

$$= AC \times BD. \tag{5.121}$$

If $AC \,//\, BD$ (as in Figure 5.1E), the line segment AC contains one but not both of B and D, the ratios AB/BC and AD/DC have opposite signs, the products $AB \times DC$ and $BC \times AD$ have opposite signs, $AB \times CD$ and $BC \times AD$ have the *same* sign, and (5.121) continues to hold when each of the expressions AB, CD, etc. is regarded as a positive length. If, on the other hand, A and C do not separate B and D (Figure 5.1C), all these equivalent statements are reversed: $AB \times CD$ and $BC \times AD$ have opposite signs. Now, when positive lengths are used, (5.121) tells us that the positive number $AC \times BD$ is equal to the *difference* between the positive numbers $AB \times CD$ and $BC \times AD$. Since their sum is greater than their difference, it follows that

$$AB \times CD + BC \times AD > AC \times BD.$$

This completes the proof of Theorem 5.12 in the case of collinear points.

Finally, if the four points do not all lie on one line, some set of three must form a triangle, and we can rename them (if necessary) so that this triangle is ABC and the remaining point (*possibly* lying on one side of the triangle) is D. Theorem 5.12 is now a consequence of Ptolemy's theorem (2.61, on page 42) and its converse (2.62), which tell us that

the mutual distances of four points A, B, C, D (the first three forming a triangle) satisfy

$$AB \times CD + BC \times AD \geq AC \times BD,$$

with the equals sign only when $ABCD$ is a cyclic quadrangle whose diagonals are AC and BD.

EXERCISE

1. Write down the whole set of eight symbols equivalent to $AC//BD$.

5.2 Cross ratio

Any four distinct points A, B, C, D determine a number $\{AB, CD\}$ called the *cross ratio* of the points in this order; it is defined in terms of four of their mutual distances by the formula

$$\{AB, CD\} = \frac{AC \times BD}{AD \times BC}.$$

Using this notation, we can divide both sides of the inequality in Theorem 5.12 by $AC \times BD$ to obtain

THEOREM 5.21. *The cross ratios of four distinct points A, B, C, D satisfy*

$$\{AD, BC\} + \{AB, DC\} = 1$$

if and only if $AC // BD$.

This criterion for separation in terms of cross ratios enables us to turn the tables: instead of defining separation in terms of circles, we can now define circles in terms of separation! Any three distinct points A, B, C determine a unique *circle* (or line) ABC, which may be described as consisting of *the three points themselves along with all the points X such that*

$$BC // AX \quad \text{or} \quad CA // BX \quad \text{or} \quad AB // CX.$$

EXERCISES

1. $\{AB, CD\} = \{BA, DC\} = \{CD, AB\} = \{DC, BA\}$.

2. Evaluate $\{AD, BC\} + \{AB, DC\}$ when
   (i) B and D divide the segment AC internally and externally in the same ratio, so that $AB/BC = AD/CD$,
   (ii) D is the center of an equilateral triangle ABC,
   (iii) $ABDC$ is a square,
   (iv) $ABCD$ is a square.

5.3 Inversion

The following "quasi-tranformation" was invented by J. Steiner about 1830. Given a circle ω with center O and radius k, as in Figure 5.3A, and a point P different from O, we define the *inverse* of P to be the point P', on the ray OP, whose distance from O satisfies the equation

$$OP \times OP' = k^2.$$

It follows from this definition that the inverse of P' is P: inversion (like the familiar half-turn and reflection) is of period two. Moreover, every point outside the *circle of inversion* ω has for its inverse a point inside: inversion "turns ω inside out". *The only self-inverse points are the points on* ω.

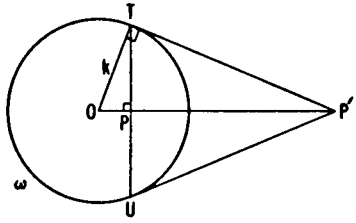


Figure 5.3A

If P describes a locus (for instance, a curve), P' describes the inverse locus. In particular, the inverse of a circle with center O and radius r is a concentric circle of radius k^2/r.

Any line through O is its own inverse, provided we omit the point O itself. (We must not try to avoid this proviso by regarding O as its own inverse, because then inversion would not be a continuous transformation; whenever P is near to O, P' is far away.)

Let P be a point inside ω (but not at O). Consider the chord TU through P, perpendicular to OP, and the point P' where the tangents at T and U intersect. Since $\triangle OPT \sim \triangle OTP'$, the point P' so constructed satisfies

$$\frac{OP}{OT} = \frac{OT}{OP'}, \qquad OP \times OP' = k^2;$$

thus it is the inverse of P.

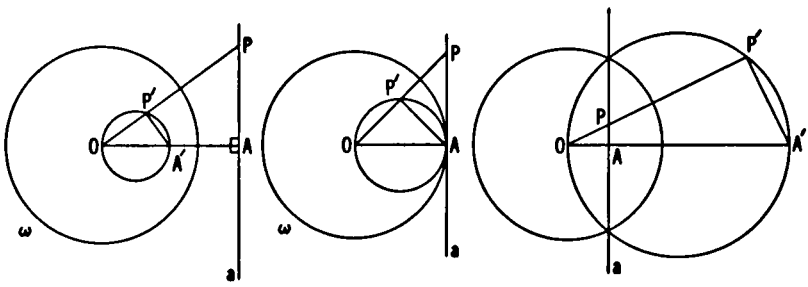


Figure 5.3B

Conversely, to construct the inverse of any point P' outside ω, we can draw the circle on OP' as diameter. If this circle intersects ω at T and U, the desired inverse P is the midpoint of TU (that is, the point where TU meets OP').

Figure 5.3B makes plausible the fact that *the inverse of any line a, not through O, is a circle through O (minus the point O itself)*, and that the diameter through O of this circle is perpendicular to a. The details are as follows. Let A be the foot of the perpendicular from O to a, let A' be the inverse of A, let P be an arbitrary point on a, and let P' be the point where the ray OP meets the circle on OA' as diameter. Then $\triangle OAP \sim \triangle OP'A'$,

$$\frac{OP}{OA} = \frac{OA'}{OP'},$$

and $OP \times OP' = OA \times OA' = k^2$.

Conversely, any point P' (except O) on the circle with diameter OA' inverts into a point P on the line a. Hence, *the inverse of any circle through O (with O omitted) is a line perpendicular to the diameter through O*, that is, a line parallel to the tangent at O to the circle.

It follows that a pair of intersecting circles, with common points O and P, inverts into a pair of intersecting lines through the inverse point P'; and that a pair of tangent circles, touching at O, inverts into a pair of parallel lines.

There is actually an instrument, not much more complicated than the compasses we use for drawing circles, which enables us to draw the inverse of any given locus. This linkage, discovered by L. Lipkin in 1781, was rediscovered by A. Peaucellier nearly ninety years later, and became known as Peaucellier's inversor, or *Peaucellier's cell.*† It consists of six rods or links: two of length a joining a fixed point O to two opposite corners Q and R of a rhombus $PQP'R$ of side b (less than a), with hinges at

† For other constructions and theory on linkages, see e.g. A. B. Kempe, *How to Draw a Straight Line*, pp. 1–51 (contained in *Squaring a Circle* by Hobson et al, Chelsea, New York, 1953); for deeper study, see I. I. Artobolevskiĭ, *Mechanisms of the Generation of Plane Curves*, Pergamon Press, 1964, or E. H. Lockwood [20].

all four corners. (See Figure 5.3C.) When a pencil point is inserted at P' and a tracing point at P (or vice versa) and the latter is traced over a given locus, the pencil draws the inverse locus. For, if X is the center of the rhombus,

$$\begin{aligned} OP \times OP' &= (OX - PX)(OX + PX) = OX^2 - PX^2 \\ &= OX^2 + RX^2 - RX^2 - PX^2 = OR^2 - PR^2 \\ &= a^2 - b^2, \end{aligned}$$

which is constant. Of course, the physical structure restricts the loci to the ring-shaped region between the circles with center O and radii $a \pm b$.

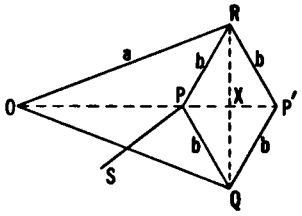


Figure 5.3C

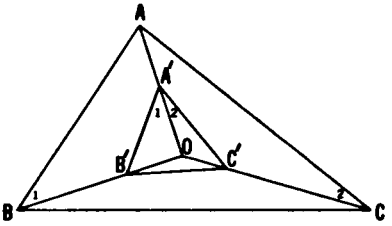


Figure 5.3D

In particular, if a seventh link SP joins P to a fixed point S whose distance from O is equal to the length of this link, P is constrained to move on a circle through O and consequently P' describes a straight line or, more precisely, a segment. Thus Peaucellier's cell solves the old problem of constructing a line without using a straightedge (whose straightness depends theoretically on the previous construction of a line).

The inverse of a triangle is usually a queer figure formed by arcs of three circles through O. Suppose, however, we restrict our attention to the *vertices* A, B, C of the triangle. If these invert into A', B', C', as in Figure 5.3D, there are some interesting relations between O, $\triangle ABC$, and $\triangle A'B'C'$. For simplicity, we suppose O to lie inside $\triangle ABC$. Since

$$OA \times OA' = k^2 = OB \times OB',$$

$\triangle OA'B' \sim \triangle OBA$, and the angles marked 1 are equal. The same is true of the angles marked 2. It follows easily that $\angle BOC$ is equal to the sum of the angles at A and A' in $\triangle ABC$ and $\triangle A'B'C'$. For, since

$$\angle BOC = \angle 1 + \angle A'B'O + \angle 2 + \angle A'C'O,$$

and since

$$\angle A'B'O = \angle BAO, \qquad \angle A'C'O = \angle CAO,$$

we have

$$\angle BOC = \angle 1 + \angle 2 + \angle BAO + \angle CAO = \angle B'A'C' + \angle BAC.$$

Similarly

$$\angle COA = \angle B + \angle B'.$$

Hence, given $\triangle ABC$, we can adjust the position of O so as to obtain a triangle $A'B'C'$ with any chosen angles A' and B'. Having found O, we can vary k and thus vary the size of $\triangle A'B'C'$ (see Exercise 6). Easy adjustments can be made if O is not inside $\triangle ABC$; it is even possible for A, B, C to be collinear. Hence

THEOREM 5.31. *For a suitable circle of inversion, any three distinct points A, B, C can be inverted into the vertices of a triangle $A'B'C'$ congruent to a given triangle.*

EXERCISES

1. Construct the inverse of a square circumscribed about the circle of inversion.

2. For what positions of O will the sides of a given triangle invert into three congruent circles?

3. Given the circle ω with center O and any point P distinct from O, construct the inverse of P using compasses only (no straightedge),†

   (i) when $OP > k/2$,

   (ii) when $\frac{k}{2n} < OP \leq \frac{k}{2(n-1)}$. [4, p. 144.]

4. How are $\triangle ABC$ and $\triangle A'B'C'$ related if O is (i) the circumcenter, (ii) the orthocenter, (iii) the incenter, of $\triangle ABC$?

5. Find coordinates for the inverse of the point (x, y) in the circle

$$x^2 + y^2 = k^2.$$

6. Given triangles ABC and DEF, sketch a construction for finding the center O and the radius k of the circle of inversion such that the inverses A', B', C' of A, B, C form a triangle congruent to $\triangle DEF$.

† It can be shown by inversion that all constructions with straightedge and compasses can be done with compasses alone; see [4, pp. 140–152], and H. P. Hudson, *Ruler and Compasses*, pp. 131–143 (contained in the aforementioned book *Squaring a Circle*).

5.4 The inversive plane

We have seen that any circle through O (with O itself omitted) inverts into a line, and that any circle with center O inverts into a circle. It is natural to ask what happens to a circle in other positions. As a first step in this direction, we proceed to find how inversion affects the distance between two points.

THEOREM 5.41. *If a circle with center O and radius k inverts a point pair AB into A'B', the distances are related by the equation*

$$A'B' = \frac{k^2 AB}{OA \times OB}.$$

For, since $\triangle OAB \sim \triangle OB'A'$ (Figure 5.4A), we have

$$\frac{A'B'}{AB} = \frac{OA'}{OB} = \frac{OA \times OA'}{OA \times OB} = \frac{k^2}{OA \times OB}.$$

From this we can easily deduce the preservation of cross ratio:†

THEOREM 5.42. *If A, B, C, D invert into A', B', C', D', then*

$$\{A'B', C'D'\} = \{AB, CD\}.$$

In fact,

$$\{A'B', C'D'\} = \frac{A'C' \times B'D'}{A'D' \times B'C'} = \frac{\dfrac{k^2 AC}{OA \times OC} \cdot \dfrac{k^2 BD}{OB \times OD}}{\dfrac{k^2 AD}{OA \times OD} \cdot \dfrac{k^2 BC}{OB \times OC}}$$

$$= \frac{AC \times BD}{AD \times BC} = \{AB, CD\}.$$

This, in turn, yields the preservation of separation:

THEOREM 5.43. *If A, B, C, D invert into A', B', C', D' and AC//BD, then A'C'//B'D'.*

For, with the help of Theorems 5.21 and 5.42, we find that the relation $AC \,//\, BD$ implies

$$\{A'D', B'C'\} + \{A'B', D'C'\} = \{AD, BC\} + \{AB, DC\} = 1,$$

whence $A'C' \,//\, B'D'$.

† J. Casey, *A Sequel to the First Six Books of the Elements of Euclid* (6th ed.), Hodges Figgis, Dublin, 1892, p. 100.

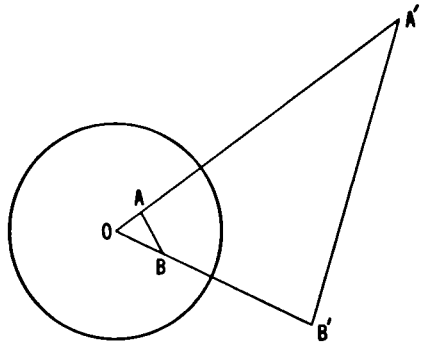


Figure 5.4A

At the end of Section 5.2 (page 107), we saw that any given circle can be described, in terms of three of its points, as consisting of A, B, C and all points X satisfying $BC \,//\, AX$ or $CA \,//\, BX$ or $AB \,//\, CX$. Hence the inverse of the given circle consists of A', B', C' and all points X' satisfying $B'C' \,//\, A'X'$ or $C'A' \,//\, B'X'$ or $A'B' \,//\, C'X'$; that is, the inverse is the circle (or line) $A'B'C'$. As we saw in Section 5.3 (page 109) the inverse is a line if and only if the given circle passes through O. This completes the proof of

THEOREM 5.44. *The inverse of any circle not passing through O is a circle not passing through O.*

The description of a circle (or line) in terms of separation suggests that it may be useful to modify our terminology so as to let the word *circle* include *line* as a special case, that is, to regard a line as a circle of infinite radius. At the same time, we agree to add to the Euclidean plane a single *point at infinity* P_∞, which is the inverse of the center of any circle of inversion. The plane, so completed, is called *the inversive plane.* Since a circle with center O inverts any circle through O into a line, we regard a line as a circle through P_∞. Since two circles tangent to each other at O invert into parallel lines, we regard parallel lines as circles tangent to each other at P_∞. With this convention, we can combine Theorem 5.44 with the results of Section 5.3 so as to obtain, for the inversive plane,

THEOREM 5.45. *The inverse of any circle is a circle.*

The addition of P_∞ to the Euclidean plane enables us to declare that inversion is a one-to-one transformation of the whole inversive plane: every point (without exception) has an inverse, and every point is the inverse of some point.

Two circles are said to be *intersecting*, *tangent* or *non-intersecting* according as their number of common points is 2, 1, or 0. Hence a pair

of circles of any one of these three types inverts into a pair of the same type (including, among pairs of "tangent circles", one circle and a tangent line, as well as two parallel lines).

EXERCISES

1. Let A be any point outside a circle ω, A' its inverse, and P a variable point on ω; then the ratio PA/PA' is constant. Conversely, if B and C divide a given line segment AA' internally and externally in a given ratio (different from 1, as in Exercise 2(i) of Section 5.2), the circle on BC as diameter is the locus of points whose distances from A and A' are in this ratio. (The locus is called the *circle of Apollonius*.)

2. Let any point on a circle ω be joined to the ends of a diameter by lines meeting the perpendicular diameter at P and P'. Then P' is the inverse of P.

3. Through any two points inside a circle, just two circles can be drawn tangent to the given circle.

4. With any three distinct points as centers, let three circles, tangent to one another at three distinct points, be drawn. (The points do not necessarily form a triangle; they may be collinear.) Then there are exactly two circles tangent to all the three circles. These two circles are nonintersecting. (They are sometimes called *Soddy's circles* [**6**, pp. 13–16] although they were described by Steiner as long ago as 1826 in the first volume of Crelle's *Journal für Mathematik*, p. 274.)

5. Give a quick proof for Theorem 5.12, using inversion [**23**, pp. 10–11].

6. The inverse, in a circle ω with center O, of a circle α through O, is the radical axis (see page 34) of ω and α.

7. When a line is regarded as a special case of a circle, is a pair of lines through one point a pair of tangent circles or a pair of intersecting circles? Explain your answer in terms of the number of points common to the two lines.

5.5 Orthogonality

From the preservation of circles it is a small step to the preservation of angles. The two supplementary angles between two intersecting circles are naturally defined as the angles between their tangents at a point of intersection. By reflection in the line of centers, it is clear that the angles

are the same at both points of intersection. To see how angles are affected by inversion in a circle with center O, let θ be one of the angles between two lines a and b through a point P, as in Figure 5.5A. We saw, in the discussion of Figure 5.3B (page 109), that the line a inverts into a circle α through O whose tangent at O is parallel to a. Similarly, b inverts into a circle β through O whose tangent there is parallel to b. Since θ is one of the angles between these tangents at O, it is one of the angles of intersection of α and β. But these circles intersect not only at O but also at P', the inverse of P. Hence the same angle θ appears at P'

The reader can easily see what changes are needed if a or b happens to pass through O. (If both lines pass through O, they invert into themselves, and the invariance of θ is immediately clear.)

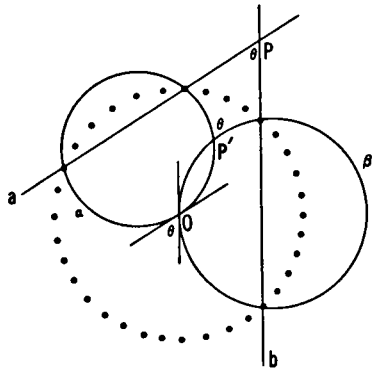


Figure 5.5A

For any two circles through P, we can let a and b be their tangents at P. The inverse circles touch α and β (respectively) at P'. Hence

THEOREM 5.51. *If two circles intersect at an angle θ, their inverses intersect at the same angle θ.*

Two circles are said to be *orthogonal* if they intersect (twice) at right angles, so that, at either point of intersection, the tangent to each is a diameter of the other. As a special case of Theorem 5.51 we have

THEOREM 5.52. *Orthogonal circles invert into orthogonal circles.*

Replacing the P of Figure 2.1B (on page 28) by O, we can regard the circle in that figure as any circle through the two inverse points A and A'. Then, since

$$k^2 = OA \times OA' = OB \times OB' = OT^2,$$

any other secant BB' through O provides another pair of inverse points,

B and B'; and either of the tangents from O has for its point of contact, T, a self-inverse point, that is, a point on the circle of inversion ω. Hence

THEOREM 5.53. *Any circle through two distinct points, inverses of each other in* ω, *is its own inverse, and is orthogonal to* ω.

Conversely, *every circle orthogonal to* ω *is its own inverse.* For, if it intersects ω at T, and A is any other point on it, the line OA meets it again at A' such that

$$OA \times OA' = OT^2 = k^2.$$

Moreover, *if two circles orthogonal to* ω *intersect, their common points are an inverse pair.* For, if A is one of these points, the line OA meets each circle again at the inverse of A.

These remarks enable us to redefine inversion in terms of orthogonality, so that we have, in fact, an "inversive" definition for inversion:

Any point on ω *is its own inverse; the inverse of any other point* P *is the second intersection of any two circles through* P *orthogonal to* ω.

Replacing ω by a line, we deduce that reflection in a line may properly be regarded as a special case of inversion in a circle.

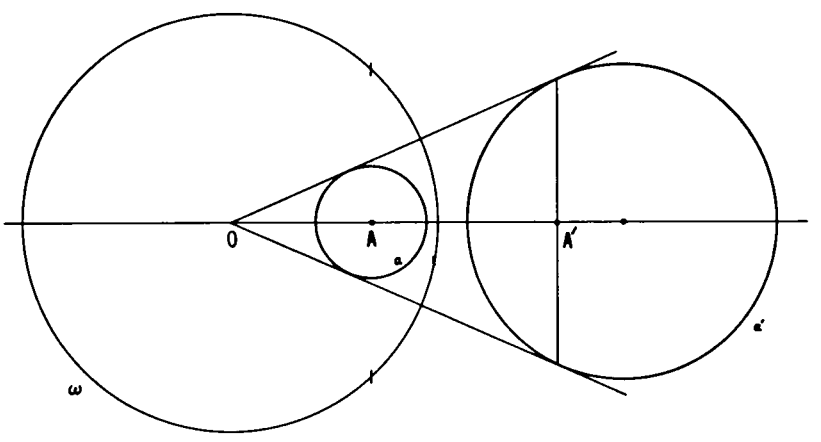


Figure 5.5B

It follows from the inversive definition of inversion that a circle α and two inverse points (inverse in α) invert (in ω) into a circle α' and two inverse points (inverse in α'). We can now combine inversive and Euclidean ideas in such a way as to discover how inversion affects the center, A, of α. We might at first expect A to invert into the center of α'; but that would be too simple! (It does not even happen

when α coincides with ω.) In fact, α and the two inverse (in α) points A and P_∞ invert (in ω) into α' and the two inverse (in α') points A' and O. Thus A' (the inverse of A in ω) is not the center of α' but the inverse (in α') of O. (See Figure 5.5B.)

EXERCISES

1. Given a circle ω and an outside point A, construct the circle with center A orthogonal to ω.

2. Given a circle ω and two non-inverse points P and Q, construct the circle through P and Q orthogonal to ω.

3. Given a point P and two circles ω_1 and ω_2 not passing through P, construct the circle through P orthogonal to both ω_1 and ω_2.

4. If ω (with center O and radius k) inverts a circle α into α', what is the relation between the powers of O with respect to α and α'?

5. For any circle α and point P on α and point O not on α, there is a unique circle through O touching α at P. (See Figure 5.5C.)

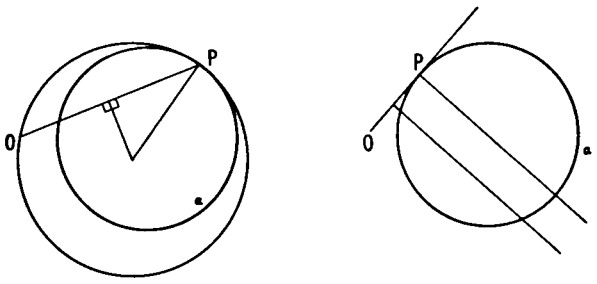


Figure 5.5C

5.6 Feuerbach's theorem

In Section 1.8 we briefly mentioned Feuerbach's theorem, to which inversion can usefully be applied in at least three ways. For one way, see Pedoe [**23**, pp. 9–10]. Before giving another [**24**, pp. 76–77], let us enunciate Feuerbach's theorem again, as follows:

THEOREM 5.61. *The nine-point circle of a triangle is tangent to the incircle and to the three excircles.*

Figure 5.6A shows triangle ABC with its medial triangle $A'B'C'$, its incircle (with center I) touching BC at X, its first excircle (with center I_a) touching BC at X_a, and the remaining common tangent B_1C_1 of these two circles (which both touch the three sides of $\triangle ABC$). We see also the circle ω on XX_a as diameter, and the points S, B'', C'' in which B_1C_1 meets BC, $A'B'$, $A'C'$. Since ω is orthogonal to the incircle and the first excircle, inversion in ω leaves both these circles invariant. We proceed to prove that ω inverts the nine-point circle $A'B'C'$ into the line B_1C_1.

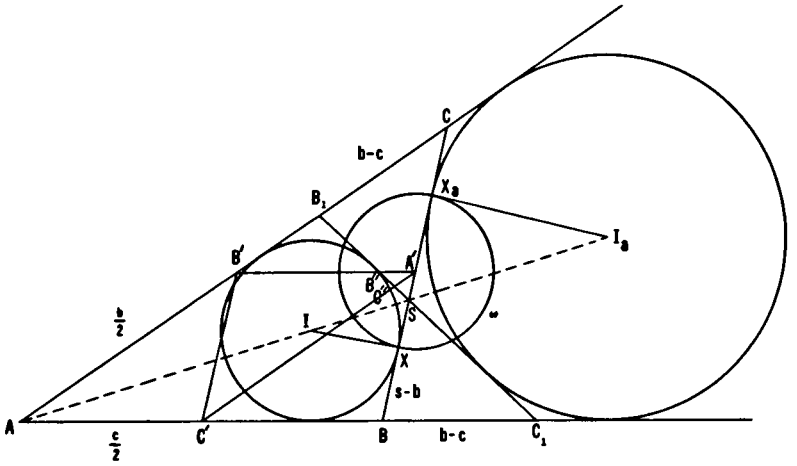


Figure 5.6A

By Theorem 1.41 (page 11) and the subsequent remarks, we have, in terms of $s = (a + b + c)/2$,

$$BX = X_aC = s - b,$$

whence the center of ω is A', the midpoint of BC, and the diameter of ω is

$$XX_a = a - 2(s - b) = b - c$$

(which we are assuming to be positive; otherwise, rename A, B, C in a different order). The nine-point circle passes through the center A' of ω; hence ω inverts it into a straight line. We shall show that this line goes through B'' and C'' (and therefore through B_1 and C_1) by showing that B'' and C'' are the inverses in ω of points B' and C' on the nine-point circle.

Since S (like I and I_a) lies on the bisector of the angle A, Theorem 1.33 (page 9) shows that S divides the segment CB (of length a) in the ratio $b : c$, so that we have

$$CS = \frac{ab}{b + c}, \qquad SB = \frac{ac}{b + c},$$

and the half-difference of these two lengths is

$$SA' = \frac{a(b-c)}{2(b+c)}.$$

Also $BC_1 = AC_1 - AB = AC - AB = b - c$, and similarly $CB_1 = b - c$.

Since $\triangle SA'B'' \sim \triangle SBC_1$ and $\triangle SA'C'' \sim \triangle SCB_1$, we have

$$\frac{A'B''}{b-c} = \frac{A'B''}{BC_1} = \frac{SA'}{SB} = \frac{b-c}{2c}$$

and

$$\frac{A'C''}{b-c} = \frac{A'C''}{CB_1} = \frac{SA'}{SC} = \frac{b-c}{2b},$$

$$A'B' \times A'B'' = \frac{c}{2}\frac{(b-c)^2}{2c} = \left(\frac{b-c}{2}\right)^2$$

and

$$A'C' \times A'C'' = \frac{b}{2}\frac{(b-c)^2}{2b} = \left(\frac{b-c}{2}\right)^2.$$

Thus ω, whose radius is $(b-c)/2$, inverts B' into B'', and C' into C'', as desired.

In fact, ω inverts the incircle and the first excircle into themselves, and their common tangent B_1C_1 into the nine-point circle. Hence the nine-point circle, like the line, touches them both, and similarly touches the remaining two excircles.

Incidentally, the nine-point circle is determined by the points D, E, F, which are the intersections of pairs of opposite sides of the orthocentric quadrangle $ABCH$ (see the end of Section 2.4 on page 39). In other words, the four triangles ABC, BCH, CAH, ABH all have the same nine-point circle. However, each of these triangles has its own set of four tritangent circles. Thus the orthocentric quadrangle determines a set of *sixteen* circles, all tangent to the circle DEF.

EXERCISES

1. In Figure 5.6A, the line B_1C_1 cuts BC at an angle $B - C$.

2. The circle ω inverts S into D (the foot of the altitude from A to BC).

5.7 Coaxal circles

In Section 2.3 (page 35) we saw that any two non-concentric circles, α and β, belong to a "pencil" $\alpha\beta$ of *coaxal circles*, such that the radical axis of α and β is also the radical axis of any two circles belonging to the pencil. Any point P on the radical axis has equal powers with respect to all the circles in the pencil. Whenever this power is positive, its square root is the length of the tangents from P to any of the circles, and these tangents serve as radii of circles with center P, orthogonal to all the circles. Any two such circles, say γ and δ (orthogonal to every circle in the pencil $\alpha\beta$), belong to a complementary pencil $\gamma\delta$, such that every circle in either pencil is orthogonal to every circle in the other. Each pencil has, for one of its members, a line, which serves as the radical axis of that pencil and the line of centers of the other, and of course these two lines are perpendicular. If we use them as coordinate axes, as in Section 2.3, the circles can be expressed as

$$x^2 + y^2 - 2ax + c = 0 \quad \text{and} \quad x^2 + y^2 - 2by - c = 0,$$

where c is fixed while a and b vary. If $c > 0$, the first pencil consists of *non-intersecting* circles, as in Figure 2.3A, and the second consists of *intersecting* circles, all passing through the *limiting points* $(\pm\sqrt{c}, 0)$, which may be regarded as degenerate members

$$(x - \sqrt{c})^2 + y^2 = 0 \quad \text{and} \quad (x + \sqrt{c})^2 + y^2 = 0$$

of the first pencil. If $c < 0$, we have the same arrangement turned through a right angle about the origin: the first pencil is intersecting and the second non-intersecting. Finally, if $c = 0$, we have two orthogonal pencils of *tangent* circles, all touching one of the axes at the origin.

The members of a non-intersecting pencil of coaxal circles occur in a natural order determined by the order of the points in which they meet the line segment joining the limiting points. This natural ordering enables us to say precisely which one of three members lies "between" the other two.

We may describe the pencil $\alpha\beta$ "inversively" as consisting of all the circles orthogonal to γ and δ, and the pencil $\gamma\delta$ as consisting of all the circles orthogonal to α and β. In other words, $\alpha\beta$ consists of all the circles orthogonal to any two distinct circles orthogonal to α and β.

If O and P are the common points of two intersecting circles γ and δ, inversion in any circle with center O yields two lines through P', the inverse of P. The circles orthogonal to these lines are a "pencil" of concentric circles with center P', and the pencil $\gamma\delta$ inverts into the diameters of these concentric circles. The same figure can be derived from any two non-intersecting circles α and β. For, we can easily find (Figure 5.7A) two intersecting circles, γ and δ, orthogonal to both α and β, namely, two circles of suitable radii whose centers lie on the radical axis of α and β. Hence

THEOREM 5.71. *Any two non-intersecting circles can be inverted into concentric circles.*

For this purpose, the circle of inversion may be any circle whose center is either of the limiting points O and P of the non-intersecting pencil $\alpha\beta$. If α precedes β in the natural order from O to P, any circle with center O (or P) will invert α into the larger (or smaller) of the concentric circles. By changing the radius of the circle of inversion without moving its center, we replace the pair of concentric circles by another pair *whose radii are in the same ratio*; for, the new inversion is equivalent to the old inversion followed by a suitable dilatation. By inverting in a circle with center P instead of O, we replace the pair of concentric circles by another pair whose radii are in the *reciprocal* ratio.

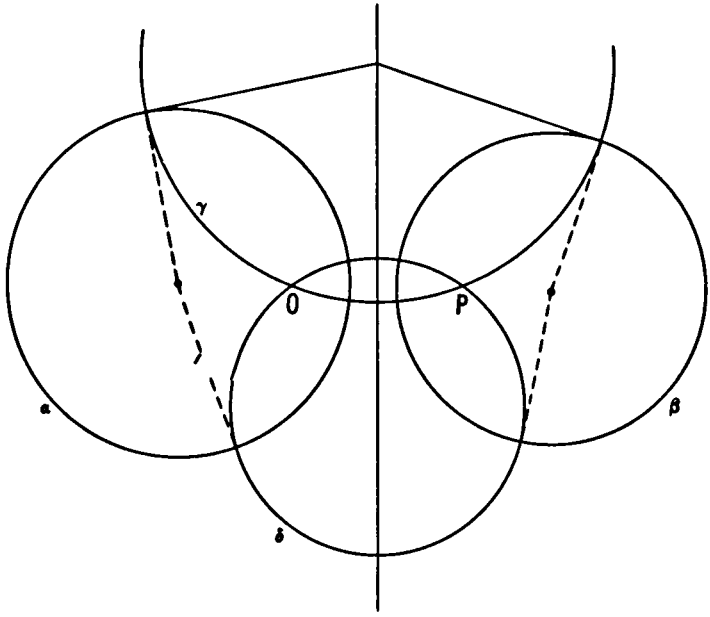


Figure 5.7A

If α and ω are any two distinct circles, the inverse of α in ω belongs to the pencil $\alpha\omega$. For, any two circles orthogonal to both α and ω invert into themselves. If α inverts into β, we call ω a *mid-circle* of α and β. (This seems more natural then the classical name "circle of antisimilitude".) Since β belongs to the pencil $\alpha\omega$, ω belongs to the pencil $\alpha\beta$. We are now ready to prove the converse of Theorem 5.45:

THEOREM 5.72. *Any two circles have at least one mid-circle. Two non-intersecting or tangent circles have just one mid-circle. Two intersecting circles have two mid-circles, orthogonal to each other.*

If α and β intersect, we can invert them into intersecting lines, which are transformed into each other by reflection in either of their angle-bisectors. Inverting back again, we see that the intersecting circles α and β have *two* mid-circles, orthogonal to each other and bisecting the angles between α and β.

If α and β are tangent, we can invert them into parallel lines. Therefore such circles have a unique mid-circle.

If α and β are non-intersecting, we can invert them into concentric circles, of radii (say) a and b. These concentric circles are transformed into each other by inversion in a concentric circle whose radius is the geometric mean $\sqrt{ab}$. Inverting back again, we see that the non-intersecting circles α and β have (like tangent circles) a unique mid-circle. If α and β are congruent, their mid-circle coincides with their radical axis.

EXERCISES

1. What equation must c and c' satisfy if the two circles

$$x^2 + y^2 - 2ax + c = 0 \quad \text{and} \quad x^2 + y^2 - 2by + c' = 0$$

are orthogonal?

2. The radius of the mid-circle of two tangent circles (on the same side of their common tangent) is the harmonic mean of the radii of the two given circles.

3. What happens when two orthogonal pencils of tangent circles are inverted in a circle whose center is their common point?

4. Any two circles can be inverted into congruent circles.

5. For any two congruent circles, their radical axis is a mid-circle.

6. Any four distinct points A, B, C, D can be inverted into the vertices of a parallelogram $A'B'C'D'$ (including, as one possibility, a degenerate parallelogram in which the four vertices lie on one line, but still $A'B' = D'C'$ and $A'D' = B'C'$). *Hint*: Consider separately the three cases (i) $AC//BD$, (ii) $AB//CD$ or $AD//BC$, (iii) A, B, C, D are not concyclic.

7. Construct the mid-circle of two given non-intersecting circles (of different sizes). *Hint*: Assume that everyone knows (with the help of Section 5.5, Exercise 3) how to locate the limiting points of the coaxal pencil $\alpha\beta$, where α and β are two non-intersecting circles (with different centers).

5.8 Inversive distance

Since angle-bisectors invert into angle-bisectors, either mid-circle of two *intersecting* circles bisects one of the *angles* between the circles. Accordingly, it is reasonable to ask whether two *non-intersecting* circles determine in some similar manner a numerical property that is bisected by their unique mid-circle. This inquiry almost forces us to invent, for any two non-intersecting circles α and β, an *inversive distance* (α, β) such that, if γ belongs to the non-intersecting pencil $\alpha\beta$ and if β lies between α and γ, then

$$(\alpha, \beta) + (\beta, \gamma) = (\alpha, \gamma). \tag{5.81}$$

Inverting in a circle whose center is one of the limiting points, we obtain three concentric circles whose radii a, b, c satisfy either $a > b > c$ or $a < b < c$ and, of course,

$$\frac{a}{b} \times \frac{b}{c} = \frac{a}{c}.$$

Noting that by taking logarithms we can transform multiplication into addition, we define

$$(\alpha, \beta) = \left| \log \frac{a}{b} \right|, \tag{5.82}$$

that is, $\log (a/b)$ or $\log (b/a)$ according as $a > b$ or $a < b$. The equation (5.81) is clearly satisfied for these concentric circles.

It would be possible to interpret the above sign "log" as meaning "logarithm to base ten", so that the relation $x = \log y$ would mean $y = 10^x$. However, the custom of using base ten arises from the non-mathematical observation that most people have ten fingers (including thumbs). It is more mathematically significant to replace this ten by the transcendental number

$$e = \sum_0^\infty \frac{1}{n!} = 2.7182818284590 \cdots,$$

so that the relation $x = \log y$ (sometimes written $\ln y$ with n for "natural") means

$$y = e^x = \sum_0^\infty \frac{x^n}{n!},$$

and the natural logarithm† itself is given by the equally remarkable series

$$\log (1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots.$$

† See [26, p. 32 ff.].

Let us agree to define the inversive distance between *any* two non-intersecting circles to be *the natural logarithm of the ratio of the radii (the larger over the smaller) of two concentric circles into which the given circles can be inverted.*

Since concentric circles invert into coaxal circles, this kind of "distance" is additive, in the sense of (5.81), for members of a coaxal pencil. In particular, *the mid-circle of any two non-intersecting circles bisects the inversive distance between them.* By regarding two parallel lines as a limiting case of two concentric circles, we see that two tangent circles may properly be considered as having inversive distance zero.

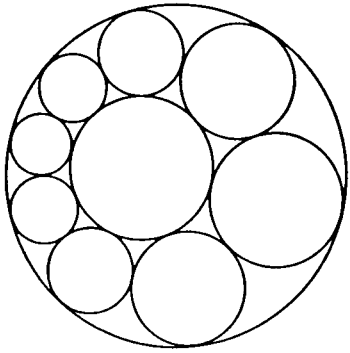


Figure 5 8A

If we have two (non-concentric) circles, one inside the other, and other circles are drawn, touching one another successively and all touching the two original circles, as in Figure 5.8A, it may happen that the sequence of tangent circles closes so as to form a ring of n, the last touching the first. In this case, we can take the first circle of the ring to be *any* circle touching both the original circles, and the ring will still close with the same value of n. Theorem 5.71 provides a remarkably simple proof of this result, known as *Steiner's porism* [**13**, p. 53]. We merely have to invert the original circles into concentric circles, and then the others become a ring of congruent circles whose centers form a regular n-gon, as in Figure 5.8B. Here A is one of the centers, T the point of contact of this circle with one of its neighbors in the ring, and O the common center of the two concentric circles: the outer one of radius a, and the inner one of radius b. $\triangle OAT$ is a right-angled triangle with

$$OA = (a+b)/2, \qquad AT = (a-b)/2$$

and angle π/n radians at O. [**8**, p. 3.] Since these concentric circles have radii a and b, their inversive distance $\delta = \log(a/b)$ satisfies

$$\sin\frac{\pi}{n} = \frac{AT}{OA} = \frac{a-b}{a+b} = \frac{(a/b)-1}{(a/b)+1} = \frac{e^{\delta}-1}{e^{\delta}+1}.$$

It follows that Steiner's porism holds whenever the inversive distance between his two original circles satisfies the same equation

$$\sin\frac{\pi}{n} = \frac{e^\delta - 1}{e^\delta + 1}.$$

Solving for e^δ and then for δ itself, we find

$$e^\delta = \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)} = \left(\frac{1 + \sin(\pi/n)}{\cos(\pi/n)}\right)^2 = \left(\sec\frac{\pi}{n} + \tan\frac{\pi}{n}\right)^2,$$

$$\delta = 2\log\left(\sec\frac{\pi}{n} + \tan\frac{\pi}{n}\right). \tag{5.83}$$

In particular, we see by setting $n = 4$ that any two circles whose inversive distance is

$$2\log(\sqrt{2} + 1)$$

belong to a "configuration" of six circles, *each* touching four others. The six circles fall into three pairs of "opposites", such that every circle touches all the others except its own opposite. The inversive distance between *any* two opposite circles is $2\log(\sqrt{2} + 1)$, and of course the remaining twelve distances are zero.

Steiner's porism is still valid if the chain of circles closes after d revolutions instead of one. In the formulae we merely have to replace n by the fraction n/d.

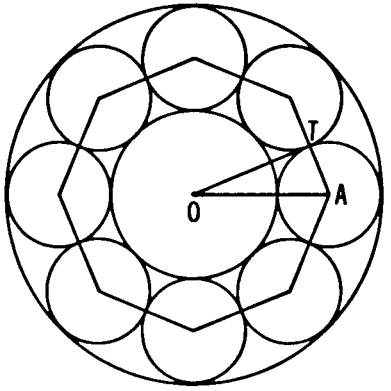


Figure 5.8B

Since a circle may have any radius, and since its center is determined by two coordinates, the set of all circles in the Euclidean plane (and also in the inversive plane) is a three-parameter family, or threefold infinity. By interpreting the threefold infinity of circles in the inversive plane as the planes of a three-dimensional space, we could obtain the famous "non-Euclidean" geometry which was discovered independently

(between 1820 and 1830) by Gauss, Bolyai and Lobachevsky. The angles between two intersecting circles appear as the angles between two planes that intersect in a line; two tangent circles appear as two "parallel" planes; and the inversive distance between two non-intersecting circles appears as the distance between two "ultraparallel" planes which have a common perpendicular line, the distance being measured along this line.†

EXERCISES

1. In Steiner's porism, the points of contact of adjacent circles in the ring lie on the mid-circle of the two original circles. (In fact, the mid-circle or mid-circles of any two circles α and β can be described as the locus of points P such that two circles, tangent to both α and β, are tangent to each other at P.)

2. Equation (5.83) is equivalent to

$$\delta = 2 \log \tan \left(\frac{\pi}{4} + \frac{\pi}{2n} \right).$$

3. Draw three congruent circles all touching one another, and a second set of three such circles, each touching also two of the first set. What are the inversive distances among these six circles?

5.9 Hyperbolic functions

In the present section we shall observe a fascinating analogy between the trigonometric functions of the angles between pairs of intersecting circles and the so-called hyperbolic‡ functions of the inversive distances between pairs of non-intersecting circles. The *hyperbolic sine*, *hyperbolic cosine* and *hyperbolic tangent* are defined, in terms of the exponential function e^x, by the formulae

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

which are easily seen to imply

$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x}.$$

† See Coxeter, *Non-Euclidean Geometry* (5th ed., Toronto, 1965), pp. 265–266.

‡ Why "hyperbolic"? See [6, p. 124], [26, p. 22]. The non-Euclidean geometry of Gauss, Bolyai and Lobachevsky is called *hyperbolic geometry*, and a nice justification for this can be seen in the article "Non-Euclidean Geometry" in *The Mathematical Sciences, A Collection of Essays* (M.I.T. Press, 1969).

Some other consequent identities are shown on the left of the following table; the column on the right gives the analogous trigonometric identities.

$\sinh 0 = 0, \quad \cosh 0 = 1$	$\sin 0 = 0, \quad \cos 0 = 1$
$\tanh 0 = 0, \quad \tanh \infty = 1$	$\tan 0 = 0, \quad \tan \frac{\pi}{4} = 1$
$\cosh^2 x - \sinh^2 x = 1$	$\cos^2 x + \sin^2 x = 1$
$\frac{\sinh x}{\cosh x} = \tanh x$	$\frac{\sin x}{\cos x} = \tan x$
$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}$	$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$
$\cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$	$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$
$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x}$	$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$

In this notation, (5.83) can be expressed as

$$\tanh \frac{\delta}{2} = \sin \frac{\pi}{n} \quad \text{or} \quad \sinh \frac{\delta}{2} = \tan \frac{\pi}{n} \quad \text{or} \quad \cosh \frac{\delta}{2} = \sec \frac{\pi}{n}.$$

It is, perhaps, not too fanciful to compare the role of the hyperbolic functions in mathematics with the role of the ammonium radical NH_4 in chemistry.† This radical behaves like an atom of sodium or potassium although it can be analyzed into atoms of nitrogen and hydrogen. Somewhat analogously, the hyberbolic functions behave like trigonometric functions although they can be expressed in terms of exponentials. (It must be admitted that this excursion into chemistry will have less appeal for any reader who, having studied functions of a complex variable, understands the meaning of the formulae $\cos x = \cosh ix$, $i \sin x = \sinh ix$.)

Returning to our discussion of angles and distances between pairs of circles, let us consider two circles of radii a and b such that the (ordinary) distance between their centers is c. If each of a, b, c is less than the sum of the other two, the circles intersect at two points, either of which forms with the two centers a triangle whose sides are a, b, c. One of the two supplementary angles of intersection, being

† A. E. H. Tutton, *Crystals* (Kegan Paul, London, 1911), p. 82.

equal to the angle between the first and second sides, has the familiar expression

$$\frac{a^2 + b^2 - c^2}{2ab}$$

for its cosine.

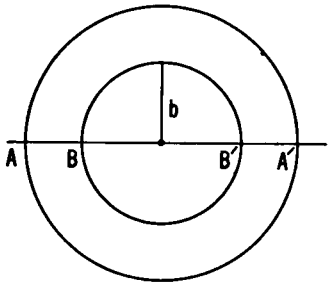


Figure 5.9A

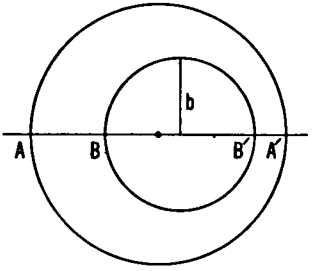


Figure 5.9B

Let us see whether we can find a geometrical meaning for the same expression

$$\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

when one of a, b, c is greater than the sum of the other two, so that the circles are non-intersecting. For instance, they might be two concentric circles (so that $c = 0$) whose diameters AA' and BB' satisfy $AB' \,//\, A'B$ on one line, as in Figure 5.9A. In terms of the inversive distance

$$\delta = \log (a/b),$$

we find the cross ratio

$$\{AA', BB'\} = \frac{AB \times A'B'}{AB' \times A'B} = \left(\frac{AB}{AB'}\right)^2 = \left(\frac{a-b}{a+b}\right)^2 = \left(\frac{e^\delta - 1}{e^\delta + 1}\right)^2$$

$$= \frac{e^{2\delta} + 1 - 2e^\delta}{e^{2\delta} + 1 + 2e^\delta} = \frac{e^\delta + e^{-\delta} - 2}{e^\delta + e^{-\delta} + 2} = \frac{\cosh \delta - 1}{\cosh \delta + 1}.$$

If these circles arise by inversion from two non-intersecting circles whose centers are at (ordinary) distance c, it is convenient to use the same letters a and b for the radii of the latter circles, and A, A', B, B' for the points where they cut their line of centers (with $AB' \,//\, A'B$, as before). By Theorems 5.42 and 5.43 (page 112), cross ratio and separation are invariant. Thus we still have

$$\{AA', BB'\} = \frac{\cosh \delta - 1}{\cosh \delta + 1},$$

although now we must express this cross ratio (and hence δ) in terms of the new a and b along with c. If $a - b > c$, as in Figure 5.9B, we have

$$\{AA', BB'\} = \frac{AB \times A'B'}{AB' \times A'B} = \frac{(a + c - b)(a - c - b)}{(a + c + b)(a - c + b)}$$

$$= \frac{(a - b)^2 - c^2}{(a + b)^2 - c^2}$$

$$= \frac{a^2 + b^2 - c^2 - 2ab}{a^2 + b^2 - c^2 + 2ab} = \frac{\gamma - 1}{\gamma + 1},$$

whence $\cosh \delta = \gamma$. Similarly, if $a + b < c$, as in Figure 5.9C,

$$\{AA', BB'\} = \frac{AB \times A'B'}{AB' \times A'B} = \frac{(c - a - b)(c + a + b)}{(c - a + b)(c + a - b)}$$

$$= \frac{c^2 - (a + b)^2}{c^2 - (a - b)^2}$$

$$= \frac{-a^2 - b^2 + c^2 - 2ab}{-a^2 - b^2 + c^2 + 2ab} = \frac{-\gamma - 1}{-\gamma + 1},$$

whence $\cosh \delta = -\gamma$. Collecting these results, we see that we have proved

THEOREM 5.91. *If c is the (ordinary) distance between the centers of two non-intersecting circles of radii a and b, the inversive distance δ between the circles is given by*†

$$\cosh \delta = \left| \frac{a^2 + b^2 - c^2}{2ab} \right|.$$

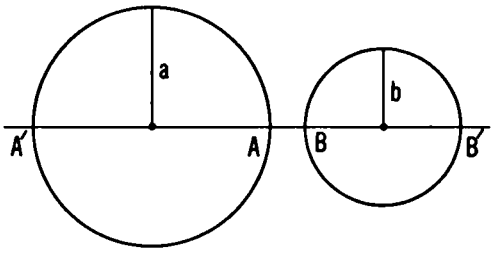


Figure 5.9C

† The graph of the function $y = \cosh x$ is the familiar *catenary*: the shape of a hanging chain supported at both ends [6, pp. 317–319].

As an interesting application of this theorem, let us consider two circles so placed that there is a quadrangle whose vertices lie on the one of radius a while its sides touch the one of radius b. It is known† that the (ordinary) distance c between the centers of two such circles satisfies the equation

$$\frac{1}{(a-c)^2}+\frac{1}{(a+c)^2}=\frac{1}{b^2},$$

which can be expressed in the form

$$|a^2+b^2-c^2| = b\sqrt{4a^2+b^2}$$

or

$$\cosh\delta = \frac{|a^2+b^2-c^2|}{2ab} = \frac{\sqrt{4a^2+b^2}}{2a} = \sqrt{1+\left(\frac{b}{2a}\right)^2}.$$

Since $\cosh^2\delta = 1+\sinh^2\delta$, it follows that the inversive distance between circles having an inscribed-circumscribed quadrangle is expressible in terms of their radii by the simple formula

$$\sinh\delta = \frac{b}{2a}.$$

EXERCISES

1. If the (ordinary) distance between the centers of two circles of radius 1 is $2(\sqrt{3}+1)$, another unit circle lying midway between them bisects their inversive distance. Is this their mid-circle?

2. The inversive distance δ between Soddy's circles (Exercise 4 of Section 5.4) is given by
$$\cosh\frac{\delta}{2} = 2.$$

3. If two circles are outside each other, so that they have four common tangents, the ratio of the lengths of the shorter and longer common tangents is $\tanh(\delta/2)$, where δ is the inversive distance between the two circles.

4. Consider a line at distance p from the center of a circle of radius b. If $p < b$, the line and the circle intersect at an angle δ given by

† [17, pp. 91–95.] According to J. L. Coolidge, *A Treatise on the Circle and the Sphere* (Oxford, 1916), pp. 45–46, it was Euler who discovered this as well as the analogous formula $1/(R-d)+1/(R+d)=1/r$ for a triangle (our Theorem 2.12 on p. 29).

$\cos \delta = \pm p/b$. If $p \geq b$, their inversive distance δ is given by $\cosh \delta = p/b$.

5. For a triangle with circumradius R and inradius r, the circumcircle and incircle are at inversive distance δ, where

$$\sinh \frac{\delta}{2} = \frac{1}{2}\sqrt{\frac{r}{R}}.$$

Hint: Use Theorem 2.12.

6. Consider the circumcircle and nine-point circle of triangle ABC. If the triangle is obtuse, these circles intersect at an angle δ given by

$$\sin^2 \frac{\delta}{2} = -\cos A \cos B \cos C.$$

If the triangle is right or acute, their inversive distance δ is given by

$$\sinh^2 \frac{\delta}{2} = \cos A \cos B \cos C.$$

7. The inversive distance between the two circles

$$x^2 + y^2 - 2ax + d^2 = 0 \quad (a > d > 0)$$

and

$$x^2 + y^2 - 2bx + d^2 = 0 \quad (b > d > 0)$$

is $|\alpha - \beta|$, where

$$\tanh \alpha = \frac{d}{a} \quad \text{and} \quad \tanh \beta = \frac{d}{b}.$$

CHAPTER 6

An Introduction to Projective Geometry

> Since you are now studying geometry and trigonometry, I will give you a problem. A ship sails the ocean. It left Boston with a cargo of wool. It grosses 200 tons. It is bound for Le Havre.... There are 12 passengers aboard. The wind is blowing East-North-East. The clock points to a quarter past three in the afternoon. It is the month of May. How old is the captain?
>
> *Gustave Flaubert*

All the transformations so far considered have taken points into points. The most characteristic feature of the "projective" plane is the principle of duality, which enables us to transform points into lines and lines into points. One such transformation, somewhat resembling inversion, is "reciprocation" with respect to a fixed circle. Every point except the center O is reciprocated into a line, every line not through O is reciprocated into a point, and every circle is reciprocated into a "conic" having O for a "focus". After some discussion of the various kinds of conic, we shall close the chapter with a careful comparison of inversive geometry and projective geometry.

6.1 Reciprocation

For this variant of inversion, we use (as in Section 5.3, page 108) a circle ω with center O and radius k. Each point P (different from O)

determines a corresponding line p, called the *polar* of P; it is the line perpendicular to OP through the inverse of P (see Figure 6.1A). Conversely, each line p (not through O) determines a corresponding point P, called the *pole* of p; it is the inverse of the foot of the perpendicular from O to p. Interchanging P and P' in Figure 5.3A, we see that, when P is outside ω, its polar joins the points of contact of the two tangents from P. Still more obviously, when P lies on ω, its polar is the tangent at P, and this is the only case in which P and p are *incident* (P on p, and p through P). We shall find it helpful to adopt a consistent notation, so that the polars of points A, B, $\cdots$ are lines a, b, $\cdots$, and the pole of any line is denoted by the corresponding capital letter.

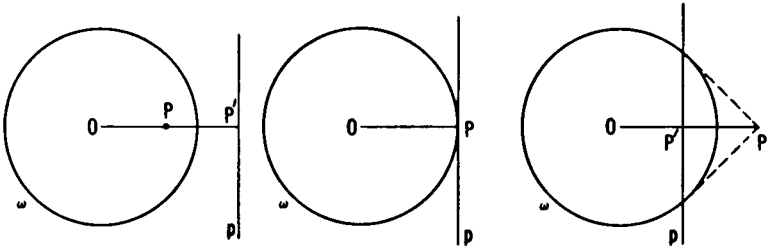


Figure 6.1A

For any point A (except O), let A' denote its inverse and a its polar, as in Figure 6.1B. For any point B on a, draw AB' perpendicular to OB. Then $\triangle OAB' \sim \triangle OBA'$, and

$$OB \times OB' = OA \times OA' = k^2.$$

Hence B' is the inverse of B, and AB' is b, the polar of B. Conversely, any line b through A (except the line OA) yields a perpendicular line OB which enables us to reconstruct the same figure. We have thus proved:

THEOREM 6.11. *If B lies on a, then b passes through A.*

By keeping A and a fixed while allowing B and b to vary, we deduce that the polars of all the points on a line a (not through O) are lines through its pole A. In other words, the polars of a set of collinear points are a set of concurrent lines. This incidence-preserving process, in which points and lines are transformed into their polars and poles, is called *reciprocation*. It leads naturally to the *principle of duality* which states that, for any configuration of points and lines, with certain points lying on certain lines, there is a dual configuration of lines and points, with certain lines passing through certain points. For instance, the dual of a

complete quadrangle $ABCD$ (consisting of four points, no three collinear, and their six joining lines AD, BD, CD, BC, CA, AB) is a complete quadrilateral $abcd$ (consisting of four lines, no three concurrent, and their six points of intersection $a \cdot d$, $b \cdot d$, $c \cdot d$, $b \cdot c$, $c \cdot a$, $a \cdot b$).

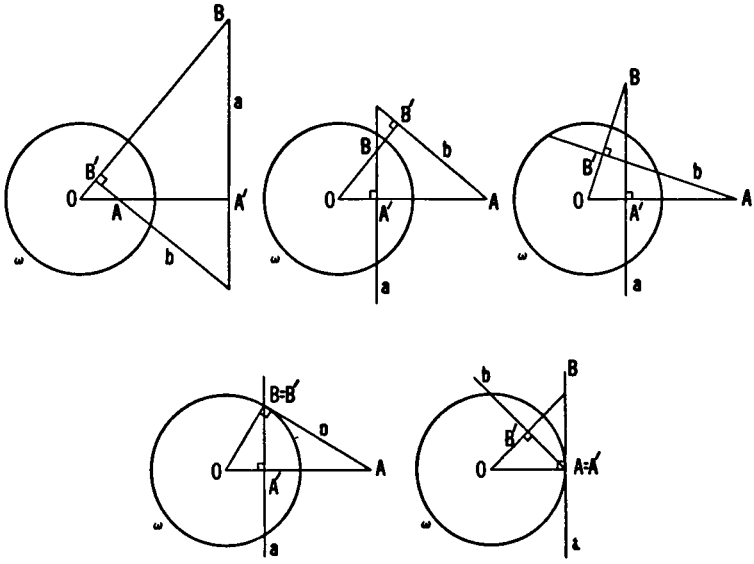


Figure 6.1B

A circle can be regarded either as a *locus* of points or as an *envelope* of lines (tangents). (See Figure 6.1C.) Each tangent is the limiting position of a secant when the two "endpoints" of the secant approach coincidence. Dually, each point of contact is the limiting position of the point of intersection of two tangents when these approach coincidence. Thus reciprocation interchanges loci and envelopes. The circle ω, regarded as a locus or an envelope, reciprocates into the same circle in the opposite aspect. Similarly, a circle with center O and radius r reciprocates (with the same change of aspect) into a concentric circle of radius k^2/r.

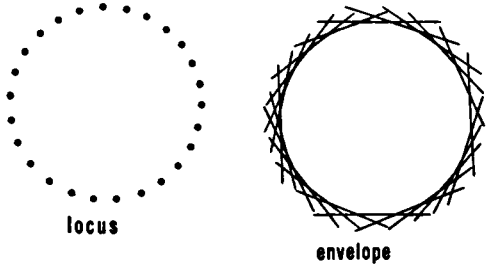


Figure 6.1C

The dual of any given theorem or construction can be obtained very simply by making certain verbal changes in accordance with the following "dictionary". (When a word in either column occurs, it must be replaced by the corresponding element in the other column.)

point	line
lie on	pass through
line joining two points	intersection of two lines
concurrent	collinear
quadrangle	quadrilateral
pole	polar
locus	envelope
tangent	point of contact

When two points and two lines are related in the manner of Theorem 6.11 (so that one lies on the polar of the other), we call A and B *conjugate points*, a and b *conjugate lines*. Thus the polar of A is the locus of points conjugate to A, and the pole of a is the envelope of lines conjugate to a. (By making the radius of a circle tend to zero, we can justify the notion that a point is the "envelope" of the lines through it.) In particular, any point on a tangent a is conjugate to the point of contact A, which is a *self-conjugate point*, and any line through A (on ω) is conjugate to the tangent a, which is a *self-conjugate line*.

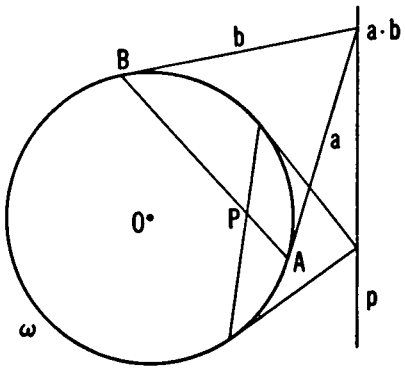


Figure 6.1D

The pole of any line AB (not through O) lies on the polars of both A and B, and thus may be described as the point of intersection $a \cdot b$. For instance, if A and B lie on ω, as in Figure 6.1D, the pole of the secant AB is the point of intersection of the tangents a and b. Dually, any point outside the circle ω lies on two tangents, say a and b, and its polar can be constructed as the secant joining the points of contact A and B.

Any line p contains some points outside ω. If p is not a diameter, its pole P lies on the polars of all these exterior points and can be con-

structed as the intersection of the polars of two of them. Dually, any point P lies on some secants. If it does not coincide with O, its polar p contains the poles of all these secants and can be constructed as the line joining the poles of two of them. We can sum up these results as follows:

THEOREM 6.12. *The pole of any secant AB (except a diameter) is the common point of the tangents at A and B. The polar of any exterior point is the line joining the points of contact of the two tangents from this point. The pole of any line p (except a diameter) is the common point of the polars of two exterior points on p. The polar of any point P (except the center) is the line joining the poles of two secants through P.*

It is worthwhile to notice that, when the reciprocating circle ω and all its tangents are given, these constructions involve only incidences of points and lines without any measurement. This feature is characteristic of projective geometry.

EXERCISES

1. With respect to a circle ω having center O, the polar of any point A (except O) can be constructed as the radical axis of two circles: ω and the circle on OA as diameter.

2. One of the angles between the polars of A and B is equal to $\angle AOB$.

3. The vertices and sides (regarded as lines) of a regular n-gon with center O reciprocate into the sides and vertices of another such n-gon.

4. A rectangle with center O reciprocates into a rhombus.

6.2 The polar circle of a triangle

Whenever the four points A, B, A', B' of Figure 6.1B are all distinct, the triangle ABC (where $C = a \cdot b$) has the property that each vertex is the pole of the opposite side, any two vertices are conjugate points, and any two sides are conjugate lines. In fact, any two conjugate (but not self-conjugate) points are two vertices of such a *self-polar triangle* ABC.

Since the three parts of Figure 6.2A (reproducing the first three parts of Figure 6.1B) are typical of every possible choice of the conjugate points A and B, *every self-polar triangle is obtuse-angled*, the vertex where the obtuse angle occurs is *inside* ω, and the remaining two vertices

are outside. Conversely, any obtuse-angled triangle ABC has a unique *polar circle* with respect to which the triangle is self-polar. Its center O and radius k can be constructed as follows. Since the lines OA and OB are two altitudes of $\triangle ABC$, O is the orthocenter. In the notation of (2.44) (page 37), the polar circle has center H and radius

$$\sqrt{HA \times HD} = \sqrt{HB \times HE} = \sqrt{HC \times HF}.$$

Therefore, inversion in this circle transforms the vertices of $\triangle ABC$ into the feet of the altitudes. Considering the circles that pass through these triads of points, and remembering that circles invert into circles, we deduce

THEOREM 6.21. *For any obtuse-angled triangle, the circumcircle and the nine-point circle are interchanged by inversion in the polar circle.*

In other words, the polar circle is one of the two mid-circles of the circumcircle and the nine-point circle. (These intersect, because the triangle is obtuse-angled.) It follows that the circumcircle, nine-point circle and polar circle (whose centers all lie on the Euler line) are *coaxal*, and that (for any obtuse-angled triangle) the nine-point circle passes through not only nine but *eleven* notable points, the last two being the points of intersection of the circumcircle and the polar circle.

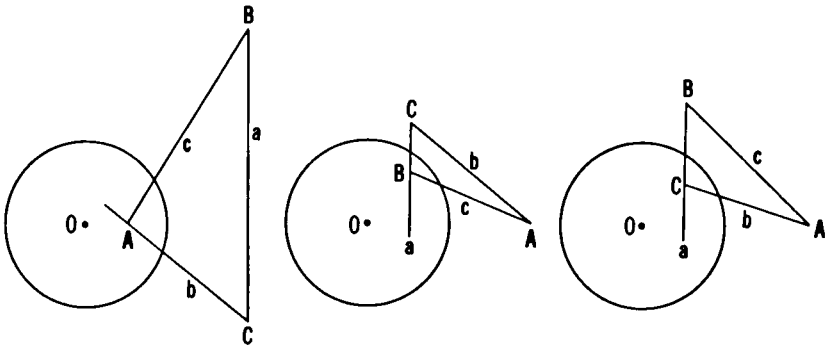


Figure 6.2A

EXERCISE

1. In an obtuse-angled triangle, the polar circle cuts the circumcircle at an angle θ such that

$$\cos^2\theta = -\cos A \cos B \cos C.$$

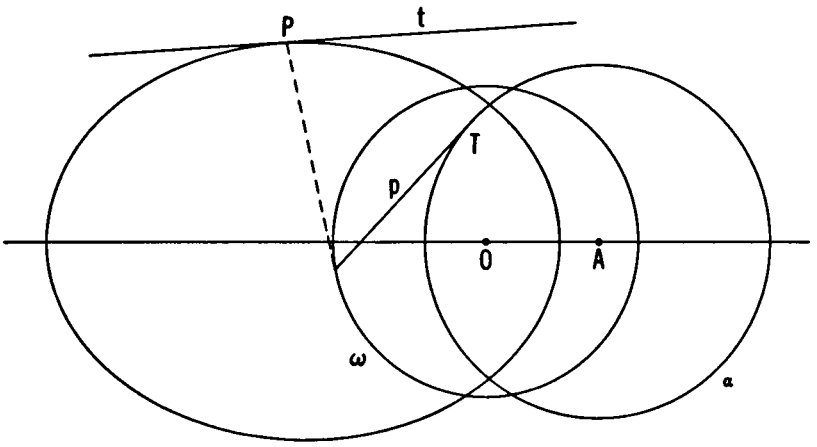


Figure 6.3A

6.3 Conics

The interesting curves called *conics* (or "conic sections"), which were mentioned briefly in Sections 3.8 and 3.9, may be approached in many different ways. One way is to *define* a conic as *the reciprocal of a circle.* More precisely, let us consider the reciprocal of a circle α, having radius r and center A, with respect to a circle ω having center O. The radius

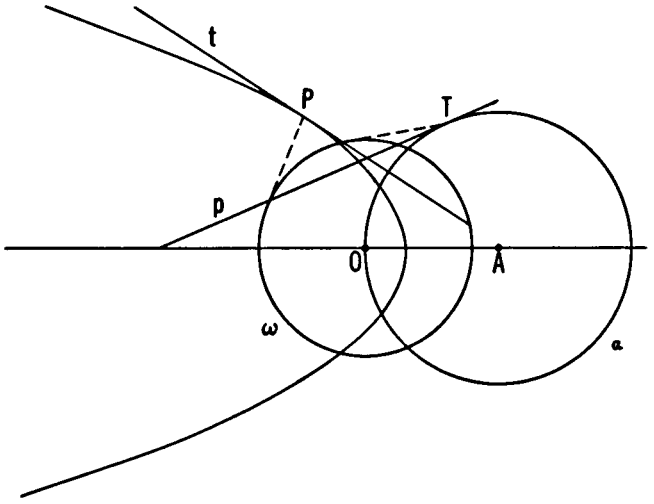


Figure 6.3B

k of ω is unimportant, as it affects only the *size* and not the *shape* of the conic. The shape is determined by the ratio

$$\epsilon = OA/r,$$

which is very naturally called the *eccentricity* of the conic. The point O is called a *focus*.

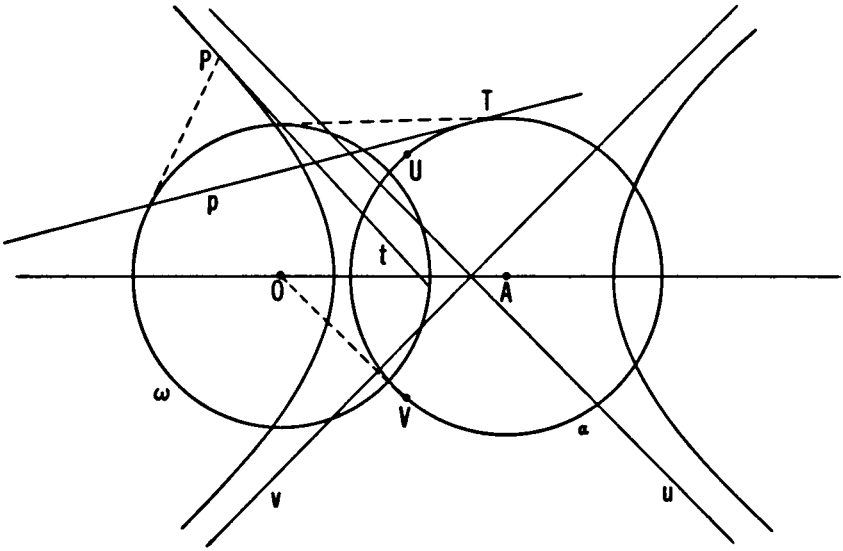


Figure 6.3C

In describing a conic as the reciprocal of α, we mean that it is both the locus of poles of the tangents to α and also the envelope of polars of the points on α. If $\epsilon < 1$, so that O is inside α, there is a point of the conic on every ray from O and the conic is an oval curve called an *ellipse* (Figure 6.3A). In particular, an ellipse with $\epsilon = 0$ is merely a circle. As the eccentricity ϵ increases, the conic becomes more and more obviously different from a circle. If $\epsilon = 1$, so that $OA = r$ and O is on α, the set of points on α includes one, namely O, which has no polar (with respect to ω), and the set of tangents to α includes one, namely the tangent at O, which has no pole; consequently the conic, which is now called a *parabola* (Figure 6.3B) extends to infinity in the direction AO. A conic is called a *hyperbola* (Figure 6.3C) if $\epsilon > 1$, so that O is outside α. The two tangents to α that pass through O have no poles; but their points of contact, U and V, have polars which are called the *asymptotes* of the hyperbola. These two lines u and v belong to the envelope and are thus tangents that have no points of contact! When we go along one of them in either direction, we see the curve getting closer

and closer without ever actually reaching the asymptote.

Sir Isaac Newton (1642–1727) explained Kepler's observation that the orbit of a planet is an ellipse having a focus in the sun. Since his time, the eccentricities ϵ for the orbits of various planets and comets have been measured. Some of these values of ϵ are given in the following table.

Planets		Comets	
Mercury	0.2056	Encke	0.85
Venus	0.0068	Biela	0.76
Earth	0.0167	Holmes	0.41
Mars	0.0934	Brooks	0.47
Jupiter	0.0484	Halley	0.967
Saturn	0.0557	Donati	0.9963
Uranus	0.0472	Coggia	0.9988
Neptune	0.0086	Daniel	1.000
Pluto	0.2481	Morehouse	1.000

EXERCISES

1. Draw two circles α and β, with nearly equal radii and nearly coincident centers, so that α lies inside β. Choose points A_1, A_3, A_5, $\cdots$ on α, and B_0, B_2, B_4, $\cdots$ on β, so that the lines B_0B_2, B_2B_4, $\cdots$ touch α at A_1, A_3, $\cdots$. Let b_2, b_4, $\cdots$ denote the lines A_1A_3, A_3A_5, $\cdots$, and let C_1, C_3, $\cdots$ be the points of intersection of the tangents to β at B_0 and B_2, B_2 and B_4, $\cdots$. Then the lines b_2, b_4, $\cdots$ are tan-

gents to the reciprocal of β with respect to α, and the points C_1, C_3, $\cdots$ lie on the reciprocal of α with respect to β.

2. The reciprocal of a circle α with respect to a non-concentric circle ω is symmetrical by reflection in the line of centers. Is it conceivable that the conic might have a second line of symmetry?

3. For a parabola, the feet of the perpendiculars from the focus to the tangents all lie on one line.

4. The angle θ at which either asymptote of a hyperbola cuts the line OA is given by $\sec\theta = \epsilon$. Deduce the eccentricity of the *rectangular* hyperbola, whose asymptotes are at right angles.

5. What happens to a comet for whose orbit $\epsilon \geq 1$?

6.4 Focus and directrix

When a conic is regarded as the reciprocal of a circle whose center is A, the polar of A (with respect to the reciprocating circle ω) is called the *directrix* (corresponding to the focus O) of the conic. For any point on a conic, the distance from a focus to the point is called a *focal distance*. We proceed to establish one of the most famous properties of a conic (proved by Pappus of Alexandria in the fourth century A.D., but possibly anticipated by Euclid six hundred years earlier):

THEOREM 6.41. *For any point P on a conic with eccentricity ϵ, focus O and directrix a, the focal distance OP is equal to ϵ times the distance from P to a.*

In Figures 6.4A, B, C, the point P is the pole (with respect to ω) of a line p which touches α at T, meets the line OA at M, and meets the line OP at P' (the inverse of P). The directrix a and the polar of M meet the line OA at A' (the inverse of A) and M' (the inverse of M); also K is the foot of the perpendicular from P to a. We wish to prove that $OP = \epsilon PK$. To cover all possible eventualities, we shall regard all distances specified on the line OA as *directed* distances

(so that $OM - OA = AM$, even if O lies between M and A). In terms of k and r, the radii of ω and α, we have

$$\frac{PK}{OP} = \frac{OA' - OM'}{OP} = \frac{k}{OP}\left(\frac{OA'}{k} - \frac{OM'}{k}\right) = \frac{OP'}{k}\left(\frac{k}{OA} - \frac{k}{OM}\right)$$

$$= \frac{OP'}{OM}\left(\frac{OM}{OA} - 1\right) = \frac{AT}{AM}\frac{AM}{OA} = \frac{r}{OA} = \frac{1}{\epsilon},$$

as desired.

Conversely,

THEOREM 6.42. *For any point O, any line a not through O, and any positive constant ϵ, the locus of a variable point whose distance from O is ϵ times its distance from a is a conic.*

This is most easily seen by taking ω to be the circle with center O that touches a, so that A is the point of contact. Then α is the circle with center A and radius OA/ϵ.

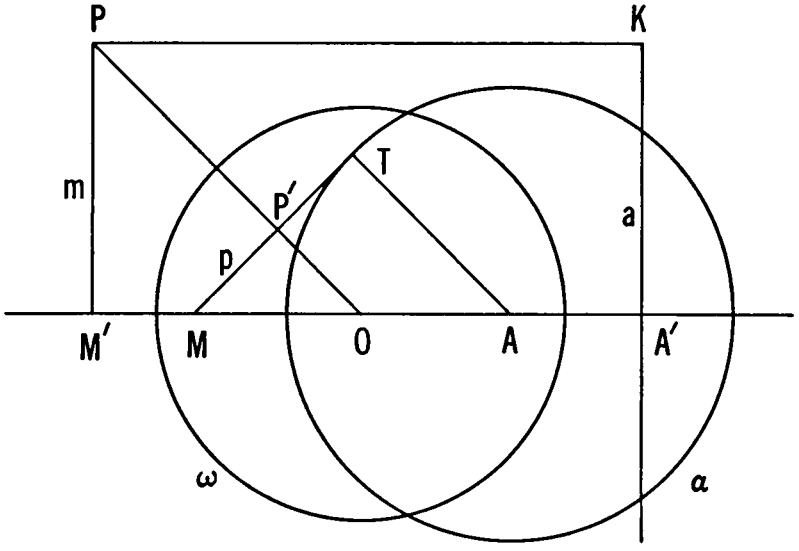


Figure 6.4A

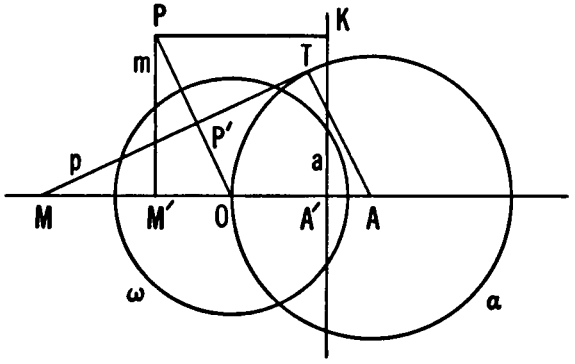


Figure 6.4B

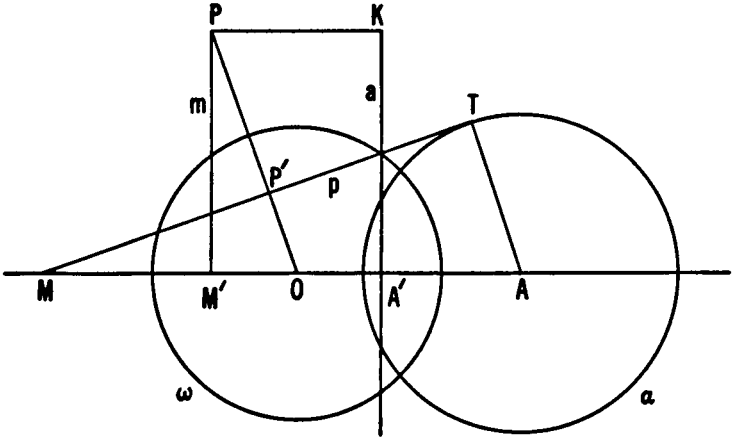


Figure 6.4C

EXERCISES

1. Obtain the Cartesian equation for the locus of a variable point P whose distance from the origin is ϵ times its distance from the line $x = l/\epsilon$.

2. If $\epsilon \neq 1$, the locus of Exercise 1 meets the x-axis twice. Shift the y-axis so as to place the new origin midway between these two meeting points. Simplify the equation by using the constants $a = l/(1 - \epsilon^2)$ and $b^2 = |la|$ instead of ϵ and l. What does the form of the equation tell us about the symmetry of the curve?

6.5 The projective plane

We can very nearly say that reciprocation transforms every point into a line, and every line into a point. The exceptions are the point O, which has no polar, and the lines through O, which have no poles. In the case of inversion, we took care of exceptions by extending the Euclidean plane into the inversive plane. In the present case, we take care of our new exceptions by a different extension: into the *projective plane.* We postulate a single *line at infinity* l_∞, which is the polar of O, and its points (the *points at infinity*) which are the poles of the lines through O. The properties of the new line and points are determined by the fact that all the points on a line a reciprocate into all the lines through its pole A. If a passes through O, the polars of its points form a "pencil" of parallel lines, namely all the lines perpendicular to a. Hence a point at infinity, such as the pole of a, has to be regarded as the common point of a pencil of parallel lines. It follows that, in the projective plane, there are no exceptions to the statement that

Any two distinct lines a and b determine a unique point $a \cdot b$.

In fact, any theorem concerning incidences of points and lines implies a dual theorem concerning lines and points, namely the polars and poles of the points and lines of the original theorem. For instance, we may take the sides of a hexagon circumscribed about the circle ω to be the tangents at the vertices of a hexagon inscribed in the same circle; thus Pascal's theorem (Section 3.8) and Brianchon's theorem (Section 3.9) are duals, and either can be deduced from the other by reciprocation with respect to ω. More generally, Pascal's theorem (or Brianchon's), applied to any circle, implies Brianchon's theorem (or Pascal's) for the reciprocal conic.

We can now simplify Theorem 6.12 by deleting the parenthetic exceptions. Moreover, when we regard this theorem as applying to an arbitrary circle α instead of the reciprocating circle ω, we can use ω to derive from α a reciprocal conic α'. Then our constructions for poles

and polars with respect to α reciprocate into constructions for "polars" and "poles" *with respect to the conic* α'. In this manner, reciprocation with respect to a circle is generalized to *polarity* with respect to a conic [6, p. 75]. Theorem 6.12 (with the parenthetic exceptions removed) consists of four parts which are duals of one another; therefore it remains true when the reciprocating circle is replaced by a conic.

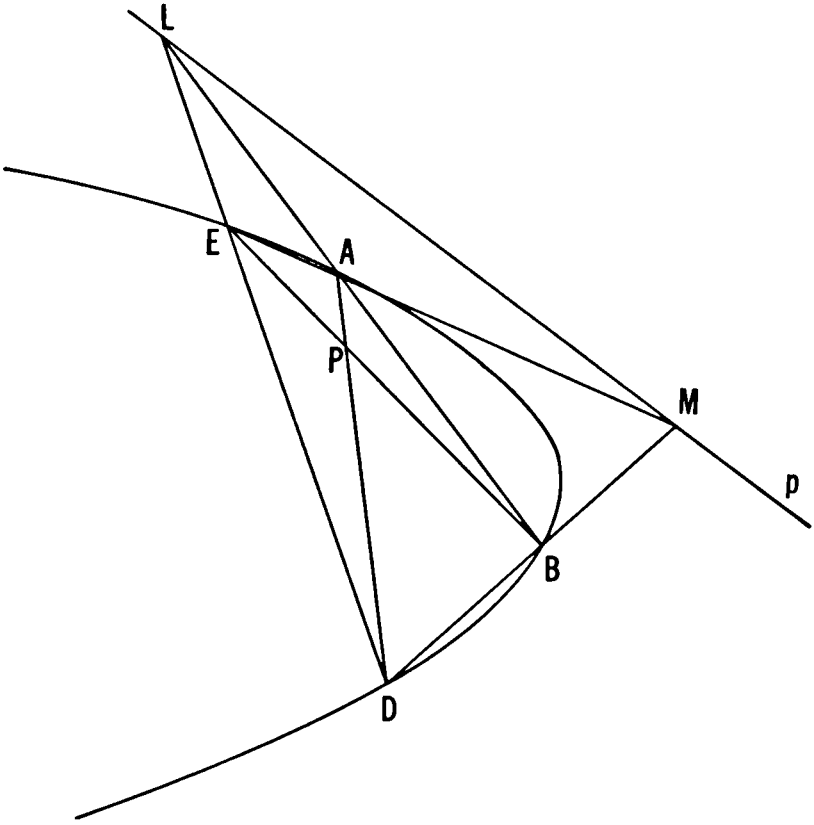


Figure 6.5A

In the notation of Figure 3.8B (page 76), the line LM passes through $N = b \cdot e$, and similarly through $a \cdot d$. This remark enables us to convert the last part of Theorem 6.12 (Figure 6.5A) into the following direct construction for the polar of a general point P:

THEOREM 6.51. *If P is not on the conic, its polar joins the points of intersection $AB \cdot DE$ and $AE \cdot BD$, where AD and BE are any two secants through P.*

We have seen that any pole and polar with respect to a circle α reciprocate (with respect to another circle ω) into a polar and pole with respect to the conic α'. In particular (see Figures 6.3A, B, C), the center A and l_∞ are pole and polar with respect to α; therefore a and O are polar and pole with respect to α':

THEOREM 6.52. *With respect to any conic except a circle, a directrix is the polar of the corresponding focus.*

EXERCISES

1. Write Theorem 3.61 (Desargues's) in its projective form, and dualize it.

2. Write Theorem 3.51 (Pappus's) in its projective form, and dualize it.

3. If a self-polar triangle for a circle has l_∞ as one of its sides, what can be said about the remaining two sides?

4. A conic is an ellipse, a parabola, or a hyperbola according as l_∞ is a non-secant, a tangent, or a secant.

5. The asymptotes of a hyperbola are its tangents at the points where it meets l_∞.

6. For a parabola, the two tangents from any point on the directrix are perpendicular.

7. For any conic through the four vertices of a complete quadrangle, the points of intersection of the three pairs of "opposite" sides are the vertices of a self-polar triangle.

6.6 Central conics

It is natural to wonder whether ellipses and hyperbolas are really more symmetrical than our constructions would immediately lead us to expect: whether the two "ends" of an ellipse are alike, and whether the two disconnected "branches" of a hyperbola are alike. The following discussion will be seen to yield the desired extra symmetry.

Revising the notation of Theorem 6.51, we can assert that, if a point C is not on the conic, its polar joins the points of intersection $PQ \cdot P_1Q_1$ and $PQ_1 \cdot P_1Q$, where PP_1 and QQ_1 are any two secants through C. If the polar of C is the line at infinity, as in Figure 6.6A, this means

that the inscribed quadrangle PQP_1Q_1 is a parallelogram. Since C is not on the conic, its polar l_∞ is not a tangent, and the conic is not a parabola. Since the diagonals of a parallelogram bisect each other, this point C (which is the pole of l_∞) is the mid-point of each of the segments PP_1, QQ_1. But these may be any two chords through C. Accordingly, C is called the *center* of the conic, ellipses and hyperbolas are called *central conics*, and we have proved

THEOREM 6.61. *A central conic is symmetrical by the half-turn about its center.*

By applying the half-turn about C to the focus O and directrix a (Section 6.4), we obtain a second focus O_1 and a second directrix a_1, as in Figures 6.6B, C. By applying the same half-turn to the circles ω and α of Section 6.3, we obtain new circles ω_1 and α_1 such that the same central conic α' is the reciprocal of α_1 with respect to ω_1.

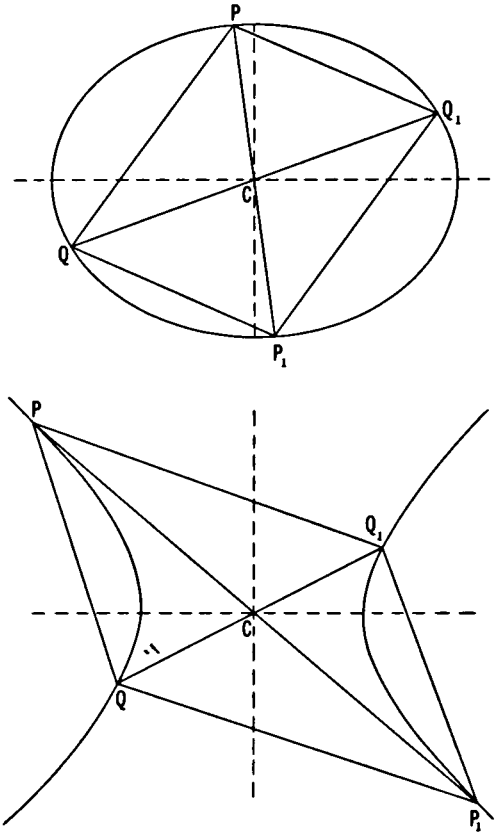


Figure 6.6A

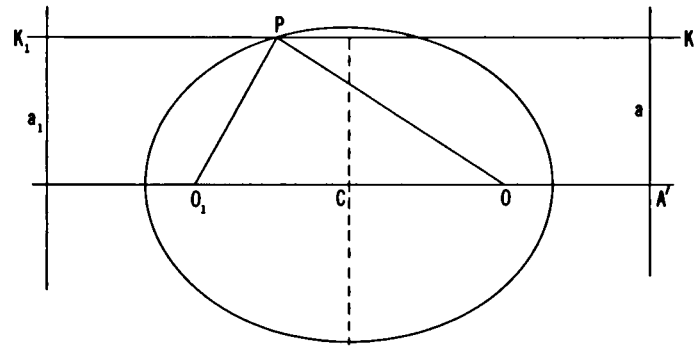


Figure 6.6B

Setting aside the trivial case when O and A coincide, we see that every conic is symmetrical by reflection in the line OA. In the case of a central conic, it follows that C lies on this line. We can express the half-turn about C as the sum of reflections in two perpendicular lines through C, one of which can be taken to be OA. Hence the central conic is also symmetrical by reflection in the line through C perpendicular to OA. In other words, the central conic has the same type of symmetry as a rhombus or a rectangle.

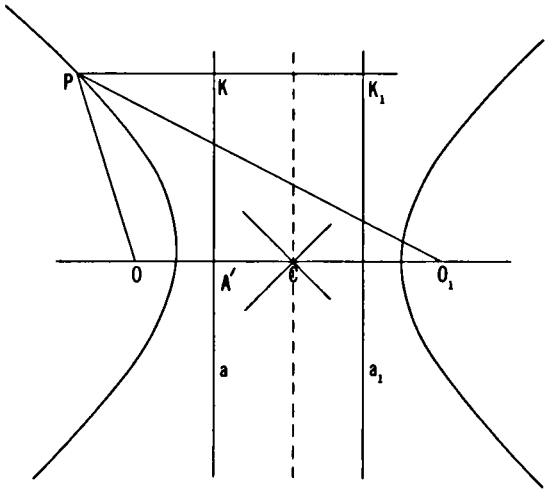


Figure 6.6C

Let c denote the polar of C with respect to ω, as in Figures 6.6D, E. Since C and l_∞ are pole and polar with respect to α', c and O must be polar and pole with respect to α. Thus C is the ω-pole of c, which is the α-polar of O. In other words, if C' is the point where c meets the line OA, C is the ω-inverse of C', which is the α-inverse of O. Since

$$OC \times OC' = k^2 = OA \times OA'$$

and

$$r^2 = AO \times AC' = OA \times C'A$$

(in the notation of directed distances), we have

$$\frac{OC}{OA'} = \frac{OA}{OC'} = \frac{OA}{OA - C'A} = \frac{OA^2}{OA^2 - (OA \times C'A)}$$

$$= \frac{OA^2}{OA^2 - r^2} = \frac{\epsilon^2}{\epsilon^2 - 1},$$

which is negative or positive according as $\epsilon < 1$ or $\epsilon > 1$. Hence, for an ellipse the center C and directrix a are on opposite sides of O, as in Figure 6.6B, but for a hyperbola they are on the same side, as in Figure 6.6C. In other words, the ellipse encloses its two foci and lies entirely between its two directrices, but the two directrices of a hyperbola both lie in the "empty" space between the two branches.

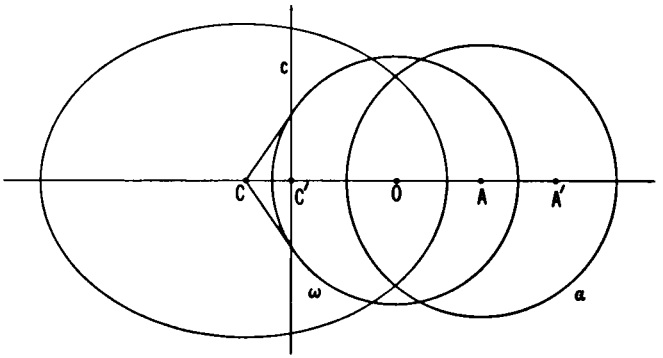


Figure 6 6D

In mechanics we learn that, when air resistance is neglected, the trajectory of a thrown ball is an arc of a parabola whose focus can be located without much difficulty. Since the thrown ball is, for a few seconds, a little artificial satellite, the apparent parabola is more accurately an enormously elongated ellipse, whose eccentricity is just a shade less than 1. Where is its second focus? At the center of the earth!

EXERCISES

1. When a point P varies on an ellipse, the sum $OP + O_1P$ of its two focal distances is constant. (See Figure 6.6B.)

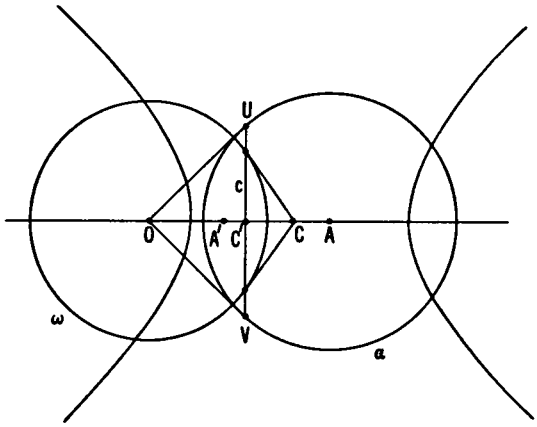


Figure 6.6E

2. When a point P varies on a hyperbola, the difference $|OP - O_1P|$ of its two focal distances is constant. (See Figure 6.6C.)

3. For a central conic, the feet of the perpendiculars from either focus to the tangents all lie on a circle. (This is called the *auxiliary circle* of the conic [20, pp. 13, 25, 155].)

6.7 Stereographic and gnomonic projection

As we saw in Section 5.3 (page 108), the only point of the Euclidean plane that has no *inverse* is the center O of the inverting circle ω. To remove this exception, and make inversion a point-to-point transformation of the whole plane, we extended the Euclidean plane by postulating a single *ideal point*, called the point at infinity, to be the inverse of O. This extended plane is called the *inversive* plane.

As we saw in Section 6.1 (page 133), the only point of the Euclidean plane that has no *polar* is the center O of the reciprocating circle ω. To remove this exception, and make reciprocation a point-to-line and line-to-point transformation of the whole plane, we extended the Euclidean plane by postulating a single *ideal line*, called the line at infinity, to be the polar of O. This extended plane is called the *projective* plane.

There are thus two different, but equally valid ways to extend the Euclidean plane. This important observation seems to be far less widely known than it should be. The two extensions can be further elucidated by working in space and comparing two of the simplest possible ways of mapping a sphere on a plane.

Our first definition for inversion in a circle (Section 5.3) is easily generalized to inversion in a sphere. Given a sphere ω with center O

and radius k, and a point P different from O, we define the *inverse* of P to be the point P', on the ray OP, whose distance from O satisfies

$$OP \times OP' = k^2.$$

By embedding the plane of Figure 5.3B (page 109) in a three-dimensional space, and rotating about the line of centers OA, we see at once that spheres (including planes as spheres of infinite radius) invert into spheres. In particular (see the middle part of Figure 5.3B), if α is the tangent plane at A to the sphere of inversion ω, then the inverse α' of α is the sphere on the radius OA as diameter. Inverse points on α and α' can actually be derived from each other without reference to ω. Given P on the plane α (see Figure 6.7A), we can construct the corresponding point P' as the second intersection of the line OP with the sphere α'. Conversely, given P', anywhere on α' *except* at O, we can construct the corresponding point P as the section of the line OP' by the plane α. Our natural desire to avoid the exception forces us to change α into an *inversive* plane by adding a single point at infinity which will be the position for P when P' is at O. [6, p. 83.]

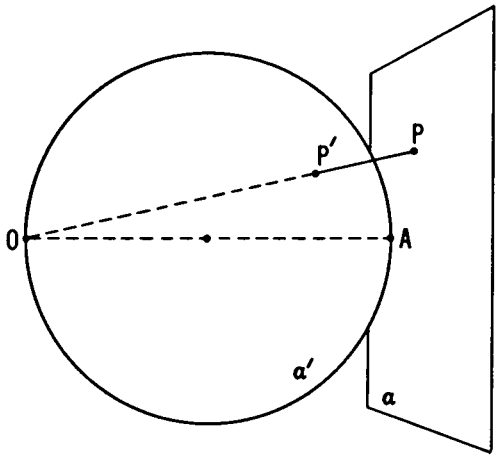


Figure 6.7A

This mapping of the sphere α' onto the plane α is called *stereographic projection*. When we notice that this kind of projection is a particular inversion, we can easily see that circles project into circles. In fact, since spheres invert into spheres (or planes), and any circle can be regarded as the curve of intersection of two spheres, it follows that circles (anywhere in space, and so, in particular, on α') invert into circles.

Another way of mapping the sphere α' onto its tangent plane α is by *gnomonic projection* (or "central projection"). Now, instead of projecting from O (antipodal to A), we project from the center of α' (which is the mid-point of OA). Since any plane through this point

meets the sphere α' in a *great circle* and the plane α in a line, each line in α comes from a great circle, and each point in α from a pair of *antipodal points* (such as P'_1 and P'_2 in Figure 6.7B) of the sphere. Conversely, given any great circle *except* the one whose plane is parallel to α, we can construct the corresponding line in α as the section by α of the plane that contains the great circle. Our natural desire to avoid the exception forces us to change α into a *projective* plane by adding a single line at infinity corresponding to the exceptional great circle. The points on this ideal line ("points at infinity") correspond to the pairs of antipodal points on the great circle. The projective statement that every two lines have a common point corresponds to the obvious fact that every two great circles have a common pair of antipodal points (i.e., that every two planes through the center of the sphere meet in a line). [**13**, p. 56.]

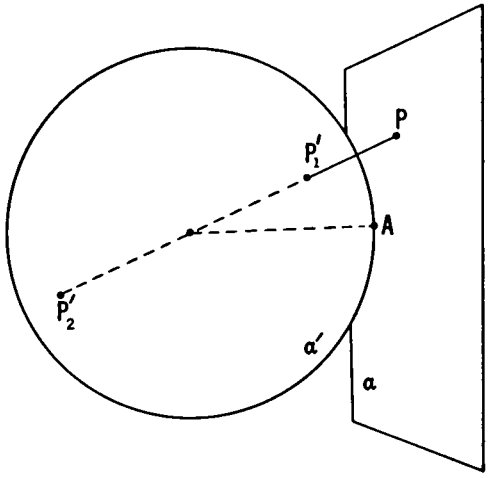


Figure 6.7B

Since all the points of the projective plane (including points at infinity) arise by gnomonic projection from pairs of antipodal points on the sphere, we can usefully regard the projective plane as being derived from the sphere by *abstractly identifying* each pair of antipodal points, that is, by changing the meaning of the word "point" so as to call such a pair *one point* [**6**, p. 94].

From the standpoint of practical map-making, neither stereographic projection nor gnomonic projection is ideal, though each has some virtues. One advantage of the former is that the angle between two directions from a point is preserved, and consequently the shapes of small islands are mapped without distortion. One advantage of the latter is that the shortest path between two points on the sphere is mapped by a straight segment.

In Theorem 5.41 (p. 112) we saw that cross ratios are preserved by inversion. Are they also preserved by reciprocation? Only in the case of *collinear* points [see **7**, pp. 118–119]. The precise statement is that *the cross ratio of four points on a line p is equal to the cross ratio of the four points at which their polars meet any line that does not pass through P, the pole of p.* The whole story is too long to be told here.

Anyone who has understood these ideas will be ready to appreciate an axiomatic treatment of projective geometry, such as [**7**]. There he will meet again the theorems of Desargues, Pappus and Pascal, from an entirely different point of view, but with the advantage of being able to recognize them as old friends.

EXERCISES

1. Stereographic projection preserves angles.

2. Stereographic projection transforms each great circle on α' into a circle (or line) in α that meets a certain circle at two diametrically opposite points of the latter.

3. If P_1', P_2' is a variable pair of antipodal points on α', and P_1, P_2 is the result of projecting stereographically, what transformation in the plane α relates P_1 to P_2?

4. Derive, by stereographic projection, the six circles of Section 5.8, Exercise 3, from the circles inscribed in the six faces of a cube.

Hints and Answers to Exercises

His answer trickled through my head
Like water through a sieve!

C. L. Dodgson

Section 1.1

1. Altitude to BC divides side a into two segments: $b \cos C$ and $c \cos B$. Add (or subtract).

2. Substitute $\sin A = a/2R$, $\sin B = b/2R$, $\sin C = c/2R$, and simplify.

3. $(ABC) = \frac{1}{2}ab \sin C$, $\sin C = c/2R$.

4. $c = 2p \sin B = pb/R$, $b = 2q \sin C = qc/R$. Multiply and simplify.

Section 1.2

1. Use Ceva with $BX = XC$, $CY = YA$, $AZ = ZB$.

2. Use Ceva with $BX = c \cos B$, $XC = b \cos C$, etc.

3. Let BB' meet CC' at O, and let OA meet $A'B'$ at A_1. Since $\triangle A'B'C' \sim \triangle ABC$,

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{OB'}{OB} = \frac{A_1B'}{AB}.$$

Therefore A_1 coincides with A'.

4. Since $\angle CXA$ and $\angle AXB$ are supplementary, the terms involving their cosines cancel out.

Section 1.3

1. The obtuse-angled triangle is inscribed in an arc smaller than a semicircle. Two of the altitudes meet their opposite sides *extended.*

2. Using Figure 1.3B, draw $A'D$ equal and parallel to BB', so that $A'CDB'$ is a parallelogram whose center E is the midpoint of CB'. Then the sides of $\triangle DAA'$ are equal and parallel to the three medians of $\triangle ABC$, and

$$\frac{(ABC)}{(DAA')} = \frac{(CAA')}{(EAA')} = \frac{CA}{EA} = \frac{4}{3}.$$

3. Let the equal medians BB' and CC' meet at G, as in Figure 1.3B. Since $BG = \frac{2}{3}BB' = \frac{2}{3}CC' = CG$, $\triangle GBC$ is isosceles and $\angle C'CB = \angle B'BC$. By the side-angle-side criterion,

$$\triangle C'CB \cong \triangle B'BC, \qquad \text{whence} \qquad B = C.$$

4. Let BE and CF be the equal altitudes. Since

$$b\,BE = 2(ABC) = c\,CF, \qquad b = c.$$

5. In the notation of Figure 1.3D, $BL/LC = c/b$, etc.

6. By Stewart's theorem (Ex. 4 of Section 1.2),

$$a(p^2 + \tfrac{1}{4}a^2) = \tfrac{1}{2}a(b^2 + c^2),$$

whence

$$p = \tfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$$

7. Use Stewart's theorem with $m = kc$, $n = kb$, $k = a/(b + c)$.

8. $12\sqrt{2}/7$.

9. Adding the altitude CF to Figures 1.1A and B, we observe that $\triangle BCJ \sim \triangle FCA$, whence $BC/CJ = FC/CA$.

Section 1.4

1. Their radii are x, y, z, in the notation of Figure 1.4A, and thus $y + z = a$, $z + x = b$, $x + y = c$. Adding gives $x + y + z = s$, etc.

2. Use Theorem 1.42 and Ex. 3 of Section 1.1.

3. Use Ceva with $AY = AZ = x$, $BZ = BX = y$, $CX = CY = z$.

4. The internal and external bisectors of angle A are at right angles.

5.
$$\begin{aligned}(ABC) &= (I_aCA) + (I_aAB) - (I_aCB)\\ &= \tfrac{1}{2}(b + c - a)r_a = (s - a)r_a.\end{aligned}$$

Alternatively, since $\triangle AI_aY_a \sim \triangle AIY$, $r_a/r = s/(s - a)$.

6. $\dfrac{r}{r_a} + \dfrac{r}{r_b} + \dfrac{r}{r_c} = \dfrac{s - a}{s} + \dfrac{s - b}{s} + \dfrac{s - c}{s} = 1.$

Section 1.5

1. Since $\angle BCM = 48° = \angle CMB$ and $\angle CBN = 12° = \angle BNC$, $BM = BC = CN$. Notice that the excenter I_a lies on the segment BM but not on the segment CN.

2. When applied to Bottema's triangle, Lemma 1.512 is, of course, true as it stands. But if we try to substitute "external" for "internal" we find that the circle BCN meets the line BM at a point M' on the side of E away from M; thus we can no longer assert that $BM > BM'$.

3. The equation $BM = CN$ implies

$$ca\left[1 - \left(\frac{b}{c + a}\right)^2\right] = ab\left[1 - \left(\frac{c}{a + b}\right)^2\right],$$

whence

$$a(a + b + c)\{(a + b + c)(a^2 + bc) + 2abc\}(b - c) = 0.$$

Section 1.6

1. Since $BCEF$ is inscribable in a circle,

$$\angle AEF = B \quad \text{and} \quad \triangle AEF \sim \triangle ABC.$$

Similarly for the other triangles.

2. Although H still lies on the internal bisector of $\angle EDF$, it lies on

the external bisectors of $\angle FED$ and $\angle DFE$.

3. See the answer to Exercise 2.

4. $\angle HAC = 90° - C$ and $\angle OAC = 90° - B$.

Section 1.7

2. Referring to Figure 1.6A, we see that $OA'^2 = R^2 - (\frac{1}{2}a)^2$. In terms of $n = GA'$, we have $AG = 2n$ and $AA' = 3n$. By Ex. 6 of Section 1.3,

$$3n = \tfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$$

Applying Stewart's theorem (Ex. 4 of Section 1.2) to $\triangle OAA'$, we obtain

$$3n(OG^2 + 2n^2) = 2nOA'^2 + nOA^2 = n(2R^2 - \tfrac{1}{2}a^2 + R^2)$$

whence

$$OH^2 = (3OG)^2 = 9R^2 - \tfrac{3}{2}a^2 - 18n^2 = 9R^2 - (a^2 + b^2 + c^2).$$

3. Assume for definiteness that $b > c$. (Otherwise interchange B and C.) By Pythagoras, $BA^2 - BD^2 = AA'^2 - DA'^2$, that is,

$$c^2 - \left(\frac{a}{2} - DA'\right)^2 = \left(\frac{b^2 + c^2}{2} - \frac{a^2}{4}\right) - DA'^2$$

and therefore $aDA' = \frac{1}{2}(b^2 - c^2)$.

4. If the Euler line is parallel to BC, it trisects AD, so that

$$OA' = AD/3.$$

Now substitute for AD and OA' the following expressions:

$$AD = b \sin C = 2R \sin B \sin C,$$

$$OA' = R \cos A = R(\sin B \sin C - \cos B \cos C).$$

Section 1.8

1. $OA' = \frac{1}{2}AH = AK$, and OA' is parallel to AK.

2. By the remark at the end of Section 1.6, EF is perpendicular to OA and to the parallel line $A'K$. Thus the diameter $A'K$ bisects the chord EF and the arc EF.

3. ΔABC is the orthic triangle of $\Delta I_aI_bI_c$.

4. Let P be the common point, and D, E, F, the points diametrically opposite to P on the circles PBC, PCA, PAB. Then PA, PB, PC, being the perpendicular bisectors of EF, FD, DE, are perpendicular also to BC, CA, AB. Since the sides of ΔABC are half as long as those of ΔDEF, the circumradius of the former is half that of the latter, that is, half the common diameter of the given circles.

5. Since DK is a perpendicular to BC, and KA' is a diameter, the circle cuts the side BC at an angle

$$\angle DKA' = \angle HKN = \angle HAO = |B - C|.$$

(See Ex. 4 of Section 1.6.)

Section 1.9

1. Extend CP to D so as to form an equilateral triangle BDP. Since $\Delta DCB \sim \Delta PCQ$, $DB/PQ = DC/PC = 1 + (DP/PC)$.

Dividing by $DB = PB = DP$, we deduce $1/PQ = (1/PB)+(1/PC)$.

2. First relax the conditions by allowing $ABCD$ to be a rectangle. Suppose, if possible, that $PD < CD$. Then $\angle CPD > 60°$, $\angle DPA < 75°$, $AD < PD < CD$. If, on the other hand, $PD > CD$, all the inequalities are reversed. In either case $ABCD$ would not be a square. Hence, if $ABCD$ *is* a square, we must have $PD = CD$.

Or: Construct $\Delta BQC \cong \Delta APB$ (see Fig. 1.9C). Then ΔBPQ is equilateral, CQ extended is perpendicular to PB and bisects it, and $CP = CB = CD$. Similarly, $DP = DC$.

3. Choose Q so as to complete parallelograms $BCPQ$ and $ADPQ$. Since

$$\angle BAQ = \alpha = \angle BPQ,$$

the four points A, B, Q, P are concyclic. Hence

$$\gamma+\varepsilon = \angle APB = \angle AQB = \angle DPC = \delta + \varepsilon,$$

so that

$$\gamma = \delta.$$

(This solution was contributed by Daniel Sokolowski.)

4 Let DF, parallel to BC, meet AB at F. Let CF meet BD at G. Since $\triangle BCG$ is equilateral, $BG = BC$. Since $\triangle CBE$ is isosceles, $BE = BC$. Hence $\triangle BGE$ is isosceles,

$$\angle BGE = 80°, \qquad \angle FGE = 40°.$$

Since $\angle EFG = 40°$, $\triangle FEG$ is isosceles and $FE = EG$. Also, $DF = DG$. Hence $\triangle GDE \cong \triangle FDE$, DE bisects $\angle FDG$, and $\angle EDB = 30°$.

5. The ends of the equal arcs are four vertices of a regular hexagon whose remaining two vertices are the midpoints of two sides of the equilateral triangle. Extending these sides by half their lengths, we obtain a larger equilateral triangle whose three sides contain alternate sides of the hexagon. The whole pattern now becomes clear.

Section 2.1

1. $-R^2$. The center.

2. A concentric circle.

3. The length of either tangent.

4. $PT^2 - PU^2 = OU^2 - OT^2 = OQ^2 - OT^2 = QT^2$.

5. $R(R - 2r) = R^2 - 2rR = d^2 \geq 0$. But $R \neq 0$. Hence $R - 2r \geq 0$.

6. The power is $d^2 - R^2 = -2rR$.

7. Writing P for A and A for X in Figure 1.2C, we have

$$BC(PA^2 + BA \times AC) = PC^2 \times BA + PB^2 \times AC,$$

that is,

$$BC(PA^2 + CA \times AB) + PB^2 \times CA + PC^2 \times AB = 0.$$

8. Trisect BC at U and V, so that $BU = UV = VC$. Since GU is parallel to AB, and GV to AC,

$$GX\left(\frac{1}{GX}+\frac{1}{GY}+\frac{1}{GZ}\right) = 1+\frac{VX}{VC}+\frac{UX}{UB} = 1+\frac{VX-UX}{VC}$$

$$= 1+\frac{VU}{UV} = 0.$$

9. 89 miles.

Section 2.2

1. The radical axis or, if the circles intersect, the radical axis minus the common chord.

2. The four midpoints all lie on the radical axis.

3. Since $\triangle PAB \sim \triangle AQB$, $\angle PBA = \angle ABQ$, Q lies on BP, and $PB/AB = AB/QB$. Since $\triangle AQB \sim \triangle ABR$, $\angle BAQ = \angle RAB$, R lies on AQ, and $AQ/AB = AB/AR$. Since

$$PB \times QB = AB^2 = AQ \times AR,$$

A and B are equidistant from the center of the circle PQR, and this circle is symmetrical by reflection in the perpendicular bisector of the segment AB. Therefore P', Q', R' all lie on this circle (and are its remaining intersections with the lines BR, AP', AP).

4. Writing the equation in the form $(x-a)^2+(y-b)^2 = a^2+b^2-c$, we see that it represents a circle if $c < a^2+b^2$.

5. Draw a circle, whose center is not on the line of centers of the given circles, cutting one of these circles at A and B, the other at C and D. From the point of intersection of the lines AB and CD, draw the line perpendicular to the line of centers. This is the radical axis.

Section 2.3

1. Let the tangent at T meet AB at O. Since $\triangle OAT \sim \triangle OTB$ and $OT = OP$,

$$\frac{TA}{TB} = \frac{OP}{OB} = \frac{OA}{OP} = \frac{OP-OA}{OB-OP} = \frac{AP}{PB}.$$

Now use the converse of Theorem 1.33.

2. The tangents to the circles from O are all equal.

Section 2.4

1. In Figure 2.4B, the points D, E, F are the midpoints of HD', HE', HF'. Hence the sides of $\triangle D'E'F'$ are parallel to those of the orthic triangle DEF.

2. $$\angle MLN = \angle MLA + \angle ALN = \angle MBA + \angle ACN$$
$$= \tfrac{1}{2}B + \tfrac{1}{2}C = \tfrac{1}{2}(B + C).$$
Similarly $\angle NML = \tfrac{1}{2}(C + A)$ and $\angle LNM = \tfrac{1}{2}(A + B)$.

Section 2.5

1. No.

2. The point diametrically opposite to B.

3. The vertices lie on their own Simson lines.

4. Draw PB, PC, C_1A_1, A_1B_1. The cyclic quadrangles A_1PB_1C and A_1BC_1P yield
$$\angle A_1B_1P = \angle A_1CP = \angle BCP = \angle C_1BP = \angle C_1A_1P,$$
$$\angle PA_1B_1 = \angle PCB_1 = \angle PBC = \angle PBA_1 = \angle PC_1A_1,$$
and $\triangle PA_1B_1 \sim \triangle PC_1A_1$.

Section 2.6

1. Use Theorems 2.61 and 2.62 with $AB = BC = AC$.

2. Draw the diagonals AC, BD, and apply Ptolemy to $PABC$ and $PDAB$. Then $PA + PC = PB\sqrt{2}$ and $PB + PD = PA\sqrt{2}$.

3. Since $\angle QPR = \angle QAR = \angle CAD = \angle ACB$
and $\angle PRQ = \angle PAQ = \angle BAC$, $\triangle PQR \sim \triangle CBA$.

By Ptolemy, $AP \times RQ + AR \times QP = AQ \times RP$. Therefore $AP \times AB + AR \times BC = AQ \times AC$.

Section 2.7

1. Let OH be the Euler line of $\triangle ABC$, and PP' a diameter of the circumcircle. By Theorem 2.72, the Simson lines of P and P' bisect HP and HP', say at M and M', respectively. Since O, M, M', N are the midpoints of PP', HP, HP', OH (Theorem 1.82), N is also the midpoint of MM'. Since $NM = \frac{1}{2}OP = \frac{1}{2}R$ is the radius of the nine-point circle (Theorem 1.81), MM' is a diameter. If the Simson lines meet at X, $\angle MXM' = 90°$ (Theorem 2.71), and X lies on the nine-point circle.

2. In an equilateral triangle the orthocenter and the circumcenter coincide.

Section 2.8

1. The proof is essentially the same as for the Butterfly theorem itself, apart from a few changes of sign.

2. Let O be the center of the circle and Q the common point of AT and BP (extended). Since OP bisects $\angle TOB$, which is twice $\angle TAB$,

$$\angle POB = \angle QAB.$$

Thus PO is parallel to QA. Since O is the midpoint of BA, P is the midpoint of BQ. Since $\triangle AHT \sim \triangle ABQ$, the midpoint of TH lies on AP.

3. Suppose $AB < AC$ (otherwise interchange B and C). Take B' on AB, and C' on AC, so that line $B'C'$ touches the incircle at Z' (diametrically opposite to X). Then $\triangle AB'C' \sim \triangle ABC$, and the incircle of $\triangle AB'C'$ touches $B'C'$ at a point X' on AX. The two incircles have "internal" common tangents of length $t' = XZ'$ and "external common tangents (which are segments of AB and AC) of length t, say. Clearly

$$B'X' = \tfrac{1}{2}(t - t') = Z'C'.$$

Similarly, if AZ' (extended) meets BC at Z,

$$BX = ZC.$$

Hence A', the midpoint of BC, is also the midpoint of XZ. But the midpoint of XZ' is I. Therefore the midpoint of XA is collinear with A' and I.
(This solution was contributed by Daniel Sokolowski.)

Section 2.9

1. The lines UX, VY, WZ bisect the angles of the equilateral triangle XYZ.

2. $A = 108°$, $B = C = 36°$.

3. The circumference of the circle is divided into three equal arcs by A, Y', Z', and the arc $Y'Z'$ is divided into three equal parts by Z and Y.

4. In the notation of Figure 2.9B,

$$\angle BZX = 60° + \alpha \quad \text{and} \quad \angle BXC = 120° + \alpha.$$

Hence

$$\frac{ZX}{\sin\beta} = \frac{BX}{\sin(60° + \alpha)}, \quad \frac{BX}{\sin\gamma} = \frac{a}{\sin(120° + \alpha)} = \frac{2R\sin 3\alpha}{\sin(60° - \alpha)}$$

and

$$ZX = \frac{2R\sin 3\alpha \sin\beta \sin\gamma}{\sin(60° + \alpha)\sin(60° - \alpha)} = \frac{4R\sin\alpha(3-4\sin^2\alpha)\sin\beta\sin\gamma}{\cos 2\alpha - \cos 120°}$$

$$= 8R\sin\alpha\sin\beta\sin\gamma.$$

5. Taking the side of $\triangle XYZ$ as unit of measurement, we have

$$BC = Y'Z' = 3, \quad BY' = CZ' = \sqrt{3},$$

$$\tan\angle CBX = \tan\angle CBZ' = \sqrt{3}/3, \quad \tan\angle ZBY' = 1/\sqrt{3},$$

$$\angle CBX = \angle ZBY' = 30°.$$

Section 3.1

1. In Figure 3.1B, $PS = QR = \frac{1}{2}BD$, so $PS + QR = BD$. Similarly, $PQ + RS = AC$.

2. Apply Ex. 6 of Section 1.3 to triangles ABC, CDA, BDX of Figure 3.1F. (It may be of interest to note that, in this theorem, "any quadrangle" can be taken to include a skew quadrangle, whose pairs of adjacent sides lie in four distinct planes.)

3. Apply Ex. 2 with $XY = 0$.

4. Use Ptolemy's theorem, 2.61.

Section 3.2

1. Observe that tangents to a circle from an external point are equal, and use Theorem 3.22 with $s = a + c = b + d$.

2. (i) 84. (ii) $4\sqrt{26}$.

3. $r = (ABC)/s = \sqrt{(s-a)(s-b)(s-c)/s}$.

4. By Ex. 5 of Section 1.4 and Ex. 3 of Section 1.1,

$$r_a + r_b + r_c - r = (ABC)\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}\right)$$

$$= \frac{(ABC)abc}{s(s-a)(s-b)(s-c)} = \frac{abc}{(ABC)} = 4R,$$

and

$$\begin{aligned}(I_aI_bI_c) &= (I_aCB) + (I_bAC) + (I_cBA) + (ABC)\\ &= \tfrac{1}{2}(ar_a + br_b + cr_c) + sr\\ &= \tfrac{1}{2}s(r_a + r_b + r_c - r) - \tfrac{1}{2}(s-a)r_a - \tfrac{1}{2}(s-b)r_b\\ &\qquad -\tfrac{1}{2}(s-c)r_c + \tfrac{3}{2}sr\\ &= \tfrac{1}{2}s\cdot 4R - \tfrac{3}{2}(ABC) + \tfrac{3}{2}(ABC)\\ &= 2sR.\end{aligned}$$

5. $K = \dfrac{abn}{4R} + \dfrac{cdn}{4R} = \dfrac{(ab+cd)n}{4R} = \dfrac{lmn}{4R}$.

6. $l = a$, $m = b$, $n = c$, $K = abc/4R$.

7. Apply Ex. 3 of Section 1.1 to the two triangles in Figure 3.2B and add results. Obtain a second expression for K by using the other diagonal l instead of n. Multiply the two expressions together and use Ptolemy's theorem, 2.61.

8. Compare the arcs into which the circle is divided by the bisectors of the angles at V and W.

9. Draw perpendiculars to P from pairs of parallel sides of the rectangle, and use Pythagoras four times. (It follows easily that P could just as well lie outside the plane of the rectangle.)

10. Let $ABCD$ be the quadrangle inscribed in a circle of diameter d, and let P be the given point on this circle. By Ex. 9 of Section 1.3, the product of the distances of P from AB and CD is

$$\frac{PA \times PB}{d}\,\frac{PC \times PD}{d} = \frac{PB \times PC}{d}\,\frac{PD \times PA}{d}$$

$$= \frac{PA \times PC}{d}\,\frac{PB \times PD}{d}.$$

Section 3.3

1. Draw diagonals CP and CQ in the squares on the first two sides BC and CA, and an isosceles right-angled triangle BAR whose hypotenuse is the third side AB. Since $\triangle PCB \sim \triangle CQA \sim \triangle BAR$, Theorems 3.33 and 3.35 are applicable.

2. (i) PO_1, QO_2, RO_3 are the perpendicular bisectors of the sides of $\triangle ABC$.

(ii) Let AO_1, BO_2, CO_3 meet the sides of $\triangle ABC$ at X, Y, Z. Then

$$\frac{BX}{XC} = \frac{(ABO_1)}{(CAO_1)} = \frac{c \sin (B + 30°)}{b \sin (C + 30°)},$$

and there are similar expressions for CY/YA and AZ/ZB, enabling us to apply the converse of Ceva's theorem.

(iii) Since $\triangle PCA \cong \triangle BCQ$, we have $PA = BQ$, and similarly $BQ = CR$. Also, $\angle PFC = \angle PBC = 60°$, similarly

$$\angle CFQ = 60°, \qquad \angle QFA = 60°$$

and, by addition, $\angle PFA = 180°$; that is, F lies on AP. It can be shown similarly that F lies on BQ, on CR, and these three lines form six angles of $60°$ at F [6, p. 22].

3. Use the converse of Ceva's theorem, as in Ex. 2 (ii).

4. Imagine Figures 3.3B and 3.3C merged. Since the six triangles BO_1N_1, CN_1O_1, CO_2N_2, AN_2O_2, AO_3N_3, BN_3O_3 are equilateral while the six triangles AN_3O_2, AO_3N_2, O_3BN_1, N_3BO_1, N_2O_1C, O_2N_1C are directly similar to $\triangle ABC$ and congruent to one another, we have

$$N_3O_2 = O_3N_2 = BN_1 = BO_1 = O_1C = N_1C = a/\sqrt{3},$$

$$N_1O_3 = O_1N_3 = CN_2 = CO_2 = O_2A = N_2A = b/\sqrt{3},$$

$$N_2O_1 = O_2N_1 = AN_3 = AO_3 = O_3B = N_3B = c/\sqrt{3}.$$

Since

$$\angle O_1BO_3 = \angle O_1BN_1 + \angle N_1BO_3 = 60° + B$$

and

$$\angle BO_3N_2 = \angle BO_3A - \angle N_2O_3A = 120° - B,$$

the quadrangle $BO_1N_2O_3$ (whose opposite sides are equal) is a parallelogram. Letting X denote the midpoint of O_2O_3, and B' the midpoint of CA (which is also the midpoint of O_2N_2), we deduce that the line XB' is parallel to O_3N_2 and BO_1. Since $BO_1 = 2XB'$, the lines O_1X and BB' meet at a point G such that $O_1G = 2GX$ and $BG = 2GB'$. But O_1X and BB' are medians of $\triangle O_1O_2O_3$ and $\triangle ABC$. Hence these two triangles have G as their common centroid. Replacing the parallelogram $BO_1N_2O_3$ by $BN_1O_2N_3$, we find similarly that G is also the centroid of $\triangle N_1N_2N_3$.

Section 3.4

1. Let AX, BY, CZ be the external bisectors. Then

$$\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = \frac{c}{b}\frac{a}{c}\frac{b}{a} = 1.$$

2. Let AX', BY' be the internal bisectors, and CZ the external bisector. Then

$$\frac{BX'}{CX'}\frac{CY'}{AY'}\frac{AZ}{BZ} = \left(-\frac{c}{b}\right)\left(-\frac{a}{c}\right)\frac{b}{a} = 1.$$

Section 3.5

1. If the two lines AC and BD are parallel, the parallelograms $ABDE$ and $CDFA$ yield $BD = AE$ and $DF = CA$, whence, by addition, $BF = CE$. Thus $EFBC$ is a parallelogram, and EF is parallel to BC. If, on the other hand, AC and BD are not parallel, let them meet at O. Since

$$OA/OB = OE/OD \quad \text{and} \quad OC/OD = OA/OF,$$

we have

$$OB \times OE = OA \times OD = OC \times OF,$$

whence $OE/OF = OC/OB$.

2. Let C and F be the points of concurrence, as in Figure 3.5A or 3.5B, and let L be the point where AB meets DE. By Pappus, L lies on MN; that is, AB, DE, NM are concurrent.

3. By Pappus, the line MN passes through the center L of the parallelogram and thus divides opposite sides into segments that are equal in pairs.

4. 9 points; 9 lines; 3 lines per point; 3 points per line.

Section 3.6

1. If two triangles, PQR and $P'Q'R'$, are perspective from O, while QR is parallel to $Q'R'$ and RP to $R'P'$, we have

$$OQ/OQ' = OR/OR' = OP/OP'.$$

Therefore PQ is parallel to $P'Q'$.

2. 10 points; 10 lines; 3 lines per point; 3 points per line.

3. (i) OQR and $P'FE$. (ii) $OQ'R'$ and PFE. (iii) ERR' and FQQ'.

4. The vertices of each pentagon lie on the sides of the other. Yes, there are altogether six such pairs of mutually inscribed pentagons. One of the remaining five is the pair $RPP'Q'D$, $EFQOR'$.

5. Let P be a vertex of a triangle PQR with Q and R on the two given lines e and f. Take D on QR extended, E on RP ex-

tended, and let DE meet QP (extended) at F. For any Q' on e, let DQ' meet f at R', and let ER' meet FQ' at P'. Then PP' is the desired line through P. If we applied the same construction to *parallel* lines e and f, we would obtain the line through P parallel to both. (For otherwise Theorem 3.62 would be contradicted.)

Section 3.7

1. Extend the lines AB, CD, EF so as to form a triangle UVW with A and B on UV, C and D on VW, E and F on WU. Since $UE = AD = FW$, we have $UF = EW = BC$. Thus $BCFU$ is a parallelogram, and CF is parallel to AB. To deal with the centroids, let X and Y be points where BE meets CF and AD, respectively. Then $CDEX$ and $BCDY$ are parallelograms, and their centers, A' and F', being the midpoints of the diagonals DX and DB, lie on a line parallel to BX and AF. Since

$$AF = BX = 2F'A',$$

the lines AA' and FF' meet at a point G such that $AG = 2GA'$ and $FG = 2GF'$. But AA' and FF' are medians of $\triangle ACE$ and $\triangle BDF$. Hence these two triangles have G as their common centroid.

2. Six.

Section 3.8

1. Let vertices A, B, C, D, E of hexagon $ABCDEF$ lie on a circle that meets AF again at F'. The three points $L = AB \cdot DE$, $M = CD \cdot FA$, $N = BC \cdot EF$ are given to be collinear, as in Figure 3.8A. Applying Pascal to the hexagon $ABCDEF'$, we see that EF', like EF, passes through the point $N = BC \cdot LM$. Hence F' coincides with F.

2. Figure 3.8B shows how Pascal's theorem applies to a degenerate hexagon $ABBDEE$. The desired result comes similarly from $AABCCE$ or $ABCCEA$.

Section 3.9

1. Use the degenerate hexagon $BQCEPF$.

2. AC, BE, QF.

3. Use the degenerate hexagon $AZBXCY$.

Section 4.1

1. Regarding the segment a in two ways as a vector, translate $\triangle ABC$ to $\triangle A'B'C'$ on the right and to $\triangle A''B''C''$ on the left. Join the points $AB \cdot A''C''$ and $AC \cdot A'B'$.

2. A tessellation of equilateral triangles, six surrounding each vertex.

Section 4.2

1. Use quarter-turns about the centers of the squares.

2. (i) Since $CX/b = a/(a+b)$,

$$\frac{CX}{XA} = \frac{CX}{b - CX} = \frac{a}{a + b - a} = \frac{a}{b}.$$

Similarly, $BY/YC = a/b$. Also

$$\frac{AH}{HB} = \frac{(CAH)}{(CHB)} = \frac{b^2}{a^2}.$$

Since now

$$\frac{BY}{YC}\frac{CX}{XA}\frac{AH}{HB} = \frac{a}{b}\frac{a}{b}\frac{b^2}{a^2} = 1,$$

the result follows by Ceva.

(ii) $\triangle ABC$ is one half of a parallelogram $ABFC$ whose center M is the midpoint of BC. Applying Exercise 1 to this parallelogram, we see that $MO_2 = MO_3$ and these lines are perpendicular. Also $MO_1 = MC$ and *these* lines are perpendicular. Hence a quarter-turn about M takes $\triangle MO_1O_2$ to $\triangle MCO_3$.

(iii) Complete the rectangle $KCGC'$ and the parallelograms $DAJA'$, $IBEB'$. Positive and negative quarter-turns about O_1, O_2, O_3 show that the six triangles $B'IB$, $C'CG$, $CC'K$, $JA'A$, DAA', BEB' are directly congruent to $\triangle ABC$. Hence the points U, V, W are the centers of the rectangle and parallelograms.

3. Consider the effect of a rotation through 60° about one vertex of the desired equilateral triangle.

Section 4.3

1. Join A to the remaining intersection of either circle with the image of the other by the half-turn about A.

2. Let O and r be the center and radius of the given circle. With centers A and O, radii r and $2r$, draw two circles meeting at O_1 and O_2. The desired line joins A to the midpoint P of OO_1 or OO_2.

3. Consider the half-turn about the midpoint of one diagonal.

Section 4.4

1. At the foot of the altitude to side AB.

2. Let AB be the base. The third vertex C must lie on a line parallel to AB, and we have to minimize $AC + CB$.

3. The mirror joins A to the midpoint of the line of centers.

Section 4.6

1. One way is $(12,0,0)$, $(7,5,0)$, $(7,0,5)$, $(2,5,5)$, $(2,1,9)$, $(11,1,0)$, $(11,0,1)$, $(6,5,1)$, $(6,0,6)$.

2. First fill the 11 oz. and 5 oz. vessels. Give one robber the vase with 8 oz. Then use the other vessels to divide the rest in accordance with the problem [16; 13, 11, 5], which can be solved in four steps.

3. Adapting the notation of Figure 1.9B, we find similar quadrangles $AC_1PB_1 \sim AB_1'P'C_1'$.

Section 4.7

1. A circle whose radius is half that of the given circle.

2. Construct a square $CBED$ externally on the side BC. The lines AD and AE meet BC at two vertices of the desired square.

Section 4.8

1. Let $\triangle AB'C'$ be any new position of the variable triangle. Since $\triangle ACC' \sim \triangle ABB'$, $\angle ACC' = \angle ABB' = \angle ABC$.

2. From the sets of congruent segments displayed in the answer to Ex. 4 of Section 3.3 (p. 166), we see that the rotation through 120° about G, which takes O_1 to O_2, O_2 to O_3, and O_3 to O_1, takes N_3 to N_2, N_2 to N_1, and N_1 to N_3. Of course, there is a similarity that transforms O_1, O_2, O_3 into N_1, N_2, N_3, respectively. However, this similarity is not direct, but *opposite*: the sum of a dilatation and a reflection [6, pp. 74–75].

Section 4.9

1. $x' = x + a, \quad y' = y + b.$

2. $x' = -x, \quad y' = y.$

3. $x' = y, \quad y' = x.$

4. $x' = -x, \quad y' = -y.$

5. $x' = kx, \quad y' = ky.$

6. $x' = -ky, \quad y' = kx.$

7. $x' = x + a, \quad y' = -y.$

8. $x' = kx, \quad y' = -ky.$

9. $x' = x^3, \quad y' = y.$

10. $x' = x, \quad y' = \begin{cases} y \text{ if } x \geq 0, \\ -y \text{ if } x < 0. \end{cases}$

Section 5.1

1. $AC // BD, \quad AC // DB, \quad CA // BD, \quad CA // DB,$
$BD // AC, \quad DB // AC, \quad BD // CA, \quad DB // CA.$

Section 5.2

1. $\{BA, DC\} = \dfrac{BD \times AC}{BC \times AD} = \dfrac{AC \times BD}{AD \times BC} = \{AB, CD\};$
similarly for the others.

2. (i) 1; (ii) 2; (iii) 3; (iv) 1.

Section 5.3

1. A flower-shaped figure consisting of four congruent semicircles (erected externally on the sides of a smaller square).

2. The incenter and excenters.

3. (i) Let the circle with center P and radius PO meet ω at points A and B. Circles through O with centers A and B meet again at the inverse of P.

(ii) Using circles we can construct, for any point P_1, a point P_2 such that $OP_2 = 2OP_1$, and similarly a point P_n such that $OP_n = nOP_1$. If $OP_1 > k/2n$, $OP_n > k/2$, and we can construct the inverse P_n' of P_n as in (i). Then the inverse P_1' of P_1 is given by $OP_1' = nOP_n'$.

4. (i) Similar to $\triangle ABC$ itself.

(ii) Similar to the orthic triangle DEF (by 2.44 on page 37).

(iii) Similar to the triangle of excenters $I_aI_bI_c$ (by Ex. 4 of Section 1.4, and Theorem 1.61).

5. $\left(\dfrac{k^2x}{x^2+y^2}, \quad \dfrac{k^2y}{x^2+y^2}\right)$.

6. Construct an isosceles triangle BO_1C with equal angles $A + D - 90°$ at B and C, and an isosceles triangle CO_2A with equal angles $B + E - 90°$ at C and A. Circles through C with centers O_1 and O_2 meet again at the desired center O. The radius k is given by

$$k^2 = \frac{OA \times OB \times DE}{AB}.$$

Section 5.4

1. Let O be the center of ω. Then

$$\triangle OAP \sim \triangle OPA' \quad \text{and} \quad PA/PA' = OA/OP,$$

which is constant.

2. Let BC be the diameter. Then $\triangle POB \sim \triangle COP'$ and

$$PO/OB = CO/OP', \quad OP \times OP' = k^2.$$

3. Let P and Q be the points inside the given circle α. Inversion in any circle with center P yields points P_∞, Q' and a circle α'. Since P_∞ is outside α', so is Q'. The two tangents from Q' to α', being two "circles" through P_∞ and Q', are the inverses of two circles through P and Q tangent to α.

4. Use a circle of inversion with its center at one of the three points of contact. The figure inverts into two parallel lines and a circle tangent to both.

5. Inversion in any circle with center A yields three points B', C', D' such that C' lies on the line segment $B'D'$ if and only if $AC \,//\, BD$. By Theorem 5.41, the "triangle inequality" $B'C' + C'D' \geq B'D'$ is equivalent to

$$\frac{BC}{AB \times AC} + \frac{CD}{AC \times AD} \geq \frac{BD}{AB \times AD},$$

that is, $AD \times BC + AB \times CD \geq AC \times BD.$

6. If ω and α intersect or touch, this is obvious. Otherwise, let ω and α have the equations $x^2 + y^2 = k^2$ and $x^2 + y^2 = ax$. By Ex. 5 of Section 5.3, the inverse of α in ω has the equation

$$\left(\frac{k^2x}{x^2 + y^2}\right)^2 + \left(\frac{k^2y}{x^2 + y^2}\right)^2 = a\left(\frac{k^2x}{x^2 + y^2}\right),$$

that is, $k^2 = ax$.

7. Intersecting. The second point of intersection is P_∞.

Section 5.5

1. It passes through the points of intersection of ω with the circle on OA as diameter.

2. It is the circle $PP'Q$, where P' is the inverse of P.

3. It is the circle PP_1P_2, where P_1 and P_2 are the inverses of P.

4. Their product is k^4.

5. Inversion in any circle with center O yields a circle α' and a point P' on α'. There is a unique line touching α' at P'. Alternatively,

inversion in any circle with center P yields a line a and a point O' not on a. There is a unique line through O' parallel to a.

Section 5.6

1. Since $\triangle AB_1C_1$ is congruent to $\triangle ABC$ by reflection in the line AS,

$$\angle BSC_1 = \angle SBA - \angle SC_1B = B - C.$$

2. By Ex. 3 of Section 1.7, $A'D = (b^2 - c^2)/2a$. We have just seen that $A'S = a(b - c)/2(b + c)$. Hence,

$$A'S \times A'D = \left(\frac{b - c}{2}\right)^2.$$

Section 5.7

1. $c + c' = 0$.

2. Let r be the radius of the mid-circle of the two tangent circles of radii a and b. Inversion in a circle whose center is the point of contact yields a line at distance $k^2/2r$, midway between two parallel lines at distances $k^2/2a$ and $k^2/2b$. Hence

$$\frac{2}{r} = \frac{1}{a} + \frac{1}{b}.$$

3. We obtain two orthogonal pencils of parallel lines, such as the lines whose equations are $x =$ constant and $y =$ constant.

4. Take O on a mid-circle. The mid-circle is then inverted into a straight line, and the inversion in it reduces to reflection.

5. Reflection in a line is a special case of inversion in a circle.

6. (i) If $AC \,//\, BD$, let γ be the circle on which the four given points lie. Let α and β be two circles orthogonal to γ, one through A and C, the other through B and D. The circles α and β intersect, say, at L and O. Any circle with center L will invert α and β into two diameters of the circle γ', making $A'B'C'D'$ a rectangle with center O'.

(ii) If $AB \,//\, CD$ or $AD \,//\, BC$, define γ, α and β as before.

But now the circles α and β are non-intersecting. Let L and O be the limiting points of the coaxal pencil $\alpha\beta$; in other words, let L and O be the points where γ meets the line joining the centers of α and β. Any circle with center L will invert α and β into two circles having the same center O'. Since $A'C'$ and $B'D'$ (on one line) are diameters of these concentric circles, $A'B'C'D'$ is a degenerate parallelogram.

(iii) If A, B, C, D are non-concyclic, they determine four distinct circles ABC, ACD, ABD, BCD. Let μ be one of the two mid-circles of ABC and ACD, namely the one that separates B and D (so that one of these points is inside and the other outside or, if μ happens to be straight, B and D are on opposite sides of it). Similarly, let ν be the mid-circle of ABD and BCD, separating A and C. The circles μ and ν intersect, say at L and O. Any circle ω with center L will invert ABC and ACD into two congruent circles $A'B'C'$ and $A'C'D'$ whose radical axis μ' separates B' and D', so that

$$\angle A'B'C' = \angle C'D'A'.$$

Similarly, ω inverts ABD and BCD into two congruent circles $A'B'D'$ and $B'C'D'$, whose radical axis ν' separates A' and C', so that $\angle D'A'B' = \angle B'C'D'$. Hence $A'B'C'D'$ is a parallelogram. [17, p. 99.]

In each case, the point pair LO is called the *Jacobian* of the two point pairs AC and BD; see Coxeter, Abh. Math. Sem. Univ. Hamburg, 29 (1966) p. 233.

7. Let the diameters of the given circles on their line of centers be AB and CD, so named that $AC \,//\, BD$. Let α and β denote the circles whose diameters are AD and BC. Let L and M be the limiting points of the coaxal pencil $\alpha\beta$. The desired mid-circle has diameter LM. (For, this circle, being orthogonal to α and β, inverts A into D, and B into C.)

Section 5.8

1. Use Ex. 4 of Section 5.7.

2. Substitute $\theta = \pi/2 + \pi/n$ in the trigonometric identity

$$\csc\theta - \cot\theta = \tan\tfrac{1}{2}\theta.$$

3. From the standpoint of inversive geometry, this arrangement of circles is simply the figure for Steiner's porism with $n = 4$. Therefore three

of the inversive distances are $2 \log (\sqrt{2} + 1)$ and the remaining twelve are zero.

Section 5.9

1. The smaller inversive distance δ is given by

$$\cosh \delta = \left| \frac{1 + 1 - (\sqrt{3} + 1)^2}{2} \right| = \sqrt{3} + 1.$$

The hyperbolic cosine of the larger inversive distance is

$$\left| \frac{1 + 1 - 4(\sqrt{3} + 1)^2}{2} \right| = 4\sqrt{3} + 7 = 2 \cosh^2 \delta - 1 = \cosh 2\delta.$$

No, the circle between cannot be the mid-circle of the others, because it is not coaxal with them.

2. Soddy's circles come from Steiner's porism with $n = 3$; hence

$$\cosh \frac{\delta}{2} = \sec \frac{\pi}{3} = 2.$$

3. The square of the ratio of lengths is

$$\frac{c^2 - (a + b)^2}{c^2 - (a - b)^2} = \frac{c^2 - a^2 - b^2 - 2ab}{c^2 - a^2 - b^2 + 2ab} = \frac{\cosh \delta - 1}{\cosh \delta + 1} = \tanh^2 \frac{\delta}{2}.$$

4. The first part is obvious from a diagram. For the second part, use Theorem 5.91 with $a = b$ and $c = 2p$:

$$\cosh 2\delta = \frac{(2p)^2 - b^2 - b^2}{2b^2} = 2\left(\frac{p}{b}\right)^2 - 1.$$

5. $2 \sinh^2 \frac{\delta}{2} + 1 = \cosh \delta = \frac{r^2 + R^2 - (R^2 - 2rR)}{2rR} = \frac{r}{2R} + 1.$

6. We see from Figure 1.3C (page 8) that

$$AH = b \cos A \csc B = 2R \cos A.$$

Using also Ex. 4 of Section 1.6 (page 18), we deduce that

$$\begin{aligned} OH^2 &= R^2 + (2R \cos A)^2 - 4R^2 \cos A \cos (B - C) \\ &= R^2(1 - 8 \cos A \cos B \cos C). \end{aligned}$$

Since $ON = \frac{1}{2}OH$, it follows that

$$\cos\delta \text{ or } \cosh\delta = \{R^2 + (\tfrac{1}{2}R)^2 - R^2(\tfrac{1}{4} - 2\cos A\cos B\cos C)\}/R^2$$
$$= 1 + 2\cos A\cos B\cos C.$$

7. Using Ex. 4 and taking the line to be the radical axis $x = 0$, we have $\cosh\alpha = a/\sqrt{a^2 - d^2}$ and $\cosh\beta = b/\sqrt{b^2 - d^2}$.

Section 6.1

1. Since ω inverts the circle on OA as diameter into the polar a, the two circles and the line belong to one coaxal pencil; that is, a is the radical axis of the circles.

2. The polars of A and B are perpendicular to OA and OB, respectively.

3. Since the reciprocal of any figure for a circle with center O is *similar* to the reciprocal of the same figure for any other circle with the same center O, we may choose ω to be the *incircle* of the given regular polygon $ABC\cdots$. Then the poles of the sides AB, BC, $\cdots$ are the mid-points of the segments AB, BC, $\cdots$, and the polars of the vertices A, B, C, $\cdots$ are the lines joining adjacent pairs of these midpoints. Similarly, if we choose ω to be the *circumcircle*, the reciprocal is the polygon obtained by drawing tangents to this circle at each vertex.

4. The poles of two opposite sides of the rectangle are equidistant from O on one line. This holds also for the other two sides, on the perpendicular line through O, with (in general) a different distance. We thus obtain a quadrangle whose diagonals bisect each other at right angles, that is, a rhombus. Alternatively, the two axes of symmetry of the rectangle intercept congruent segments of the tangents at its vertices.

Section 6.2

1. By Theorem 6.21, the polar circle bisects one of the two supplementary angles between the circumcircle and the nine-point circle, namely the one that tends to zero when the obtuse angle tends to 180°. Hence, in the notation of Section 5.9, Ex. 6, $\theta = \frac{1}{2}(180° - \delta)$.

Section 6.3

1. With respect to α, the polar of B_2 is b_2, and so on. With respect to β, the pole of the line B_0B_2 is C_1, and so on.

2. Each of the figures in this section is symmetrical about the line OA: everything that happens above this line could have been duplicated below. The appearance of Figures 6.3A and C suggests the possibility of another "mirror", perpendicular to OA, for the ellipse and hyperbola. (This will be established in Section 6.6.)

3. We see, from Figure 6.3B, that each tangent t of the parabola is the polar of a point T on the circle α. The foot of the perpendicular from O to t is the inverse of T in ω. Its locus, being the inverse of α (through O), is a straight line.

4. We see, from Figure 6.3C, that the asymptote u, being the polar of U, is perpendicular to the side OU of the right triangle OAU. Hence this triangle has angle θ at A, and

$$\sec\theta = \frac{OA}{AU} = \frac{OA}{r} = \epsilon.$$

For the rectangular hyperbola, $\theta = 45°$ and $\epsilon = \sqrt{2}$.

5. A comet with a parabolic or hyperbolic orbit would never return to the neighborhood of the sun. However, there is no conclusive evidence that such a comet has ever been seen. Although the portion of an orbit that we can observe sometimes resembles a hyperbola because of perturbation by planets (especially the massive planet Jupiter), and some elliptic orbits are so elongated as to be indistinguishable from parabolas, all the known comets (including the "non-periodic" ones that pay us one brief visit and are never seen again) are generally regarded as members of the solar system. Their speed relative to the sun is never great enough to enable them to escape into "outer space" where the attraction of some other star might be more influential than that of the sun.

Section 6.4

1. $x^2 + y^2 = (l - \epsilon x)^2$.

2. Midway between $x = l/(\epsilon \pm 1)$, we find $x = -\epsilon a$. Thus the new equation is

$$(x - \epsilon a)^2 + y^2 = [l - \epsilon(x - \epsilon a)]^2 = (a - \epsilon x)^2,$$

or

$$(1 - \epsilon^2)x^2 + y^2 = (1 - \epsilon^2)a^2 = la = \pm b^2,$$

or

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

with the upper or lower sign according as $\epsilon < 1$ or $\epsilon > 1$. Since only even powers of x and y occur, the ellipse and hyperbola are symmetrical about both the coordinate axes.

Section 6.5

1. If two triangles are perspective from a point, they are perspective from a line. If two triangles are perspective from a line, they are perspective from a point.

2. If the six vertices of a hexagon lie alternatively on two lines, the three pairs of opposite sides meet at collinear points. If the six sides of a hexagon pass alternately through two points, the three diagonals are concurrent [7, pp. 38, 90].

3. They are perpendicular lines through the center.

4. Since l_∞ is the polar of O, any point at infinity on the conic is the pole (with respect to ω) of a tangent to α that passes through O. Hence the number of points at infinity on the conic is 0, 1, or 2 according as O is inside α, on α, or outside α.

5. In the notation of Figure 6.3C, OU is the tangent to α at U; therefore one of the points at infinity on the hyperbola is the point of contact of the tangent u, and of course the other is the point of contact of v.

6. Since the directrix is the polar of A, any point on it is the pole of a diameter of α, and the tangents to the parabola from such a point are the polars of the two ends of that diameter. Since these diametrically opposite points on α subtend a right angle at O, their polars are perpendicular.

7. Each of the three "diagonal points", in turn, can be identified with the P of Theorem 6.51, and then the other two lie on its polar.

Section 6.6

1. $OP + O_1P = \epsilon PK + \epsilon K_1P = \epsilon K_1K$. This is ϵ times the distance between the two directrices.

2. When P is on the left branch of the hyperbola, as in Figure 6.6C,

$$OP - O_1P = \epsilon PK - \epsilon PK_1 = -\epsilon KK_1.$$

For the right branch the sign has to be reversed.

3. This circle is the inverse (in ω) of α. (Compare Exercise 3 of Section 6.3.)

Section 6.7

1. Stereographic projection is a special case of inversion.

2. The plane that perpendicularly bisects the diameter OA (Figure 6.7A) cuts the sphere α' along a special great circle that we naturally call the equator. Every other great circle meets this one at a pair of diametrically opposite points. A special feature of the equator is that its diameters project into diameters (of the circle in α with center A and radius $2k$).

3. We can regard P_1' and P_2' as the points of intersection of two great circles of α', one of which passes through O and A. Thus P_1 and P_2, in α, are the points of intersection of a line through A and a circle through two diametrically opposite points, say Q_1 and Q_2, of the "projected equator" (with center A and radius $2k$). Since

$$AP_1 \times AP_2 = AQ_1 \times AQ_2 = -(2k)^2,$$

P_1 and P_2 are related by an *anti-inversion*: the sum of the inversion in the projected equator and the half-turn about its center A.

4. Take α' to be the sphere that touches the twelve edges of the cube (at their midpoints), and O to be one of the points of intersection of α' with the line joining two opposite vertices. (By moving O to one of the points of intersection of α' with the line joining the centers of two opposite faces, we would obtain instead the symmetrical figure for Steiner's porism with $n = 4$.)

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Glossary

"When I use a word," Humpty Dumpty said, "it means just what I choose it to mean—neither more nor less."

C. L. Dodgson

(ABC). Area of $\triangle ABC$.

altitude of a triangle. Line segment from a vertex perpendicular to a side (or its extension).

antipodal points on a sphere. The end-points of a diameter.

asymptote to a curve. A tangent whose point of contact is at infinity.

central conic. Ellipse or hyperbola.

central dilatation. A dilatation that keeps one point fixed.

centroid of a triangle. Point of intersection of medians.

cevian. A line segment joining a vertex of a triangle to a point on the opposite side (or on its extension).

circumcenter (O) *of a triangle.* Center of its circumscribed circle.

circumcircle of a triangle. Circle circumscribed about the triangle.

circumradius (R) *of a triangle.* Radius of its circumscribed circle.

coaxal circles. Family of circles all pairs of which have the same radical axis. Alternatively, circles orthogonal to two given circles.

collineation. A transformation that takes lines into lines.

congruence. See *isometry.*

conic. Reciprocal of circle α (center A, radius r) with respect to circle ω (center O, radius k).

conjugate lines. A line a and any line through the pole of a.

conjugate points. A point A and any point on the polar of A.

cross ratio of 4 points. $\{AB, CD\} = (AC/BC)/(AD/BD)$.

cyclic quadrangle. A convex quadrangle whose vertices lie on a circle (so that opposite angles are supplementary).

dilatation. A transformation that takes each line into a parallel line. A direction-preserving similarity.

direct similarity. A collineation that preserves angles and their sense.

directrix of a conic. The polar of A with respect to ω (see definition of *conic*).

eccentricity of a conic. $\epsilon = OA/r$ (see definition of *conic*).

ellipse. Conic with eccentricity $\epsilon < 1$, so that O is inside α (see definition of *conic*).

envelope. The set of tangents of a curve.

escribed circle. See *excircle.*

Euler line of $\triangle ABC$. The line on which the orthocenter, centroid and circumcenter lie.

excenters (I_a, I_b, I_c) *of a triangle.* Centers of escribed circles of the triangle.

excircle, or escribed circle of a triangle. A circle tangent to one side of the triangle and to the extensions of the other two sides.

exradii (r_a, r_b, r_c) *of a triangle.* Radii of escribed circles of the triangle.

focus of a conic. The center O of the reciprocating circle (see definition of *conic*).

Gergonne point of $\triangle ABC$. Point of intersection of the cevians through the points of tangency of the incircle to the sides of $\triangle ABC$.

gnomonic projection. Projection of a sphere from its center onto any tangent plane.

great circle on a sphere. The section by a plane through the center.

half-turn. A rotation by 180°.

homeomorphic. Continuously transformable both ways.

hyperbola. Conic with eccentricity $\epsilon > 1$, so that O is outside α (see definition of *conic*).

asymptotes of a hyperbola. Polars of points of contact of tangents from O to α.

image of a point P by reflection in a line l. Second intersection of two circles through P whose centers lie on l.

incenter (I) *of a triangle.* Center of its inscribed circle.

incircle of a triangle. Circle inscribed in the triangle.

inradius (r) of a triangle. Radius of its inscribed circle.

inverse of a point P with respect to a circle ω. Second intersection of two circles through P orthogonal to ω.

inversive distance between two non-intersecting circles α and β. Natural logarithm of the ratio of the radii of two concentric circles into which α and β can be inverted.

inversive plane. Euclidean plane plus a single ideal point (see *point at infinity* P_∞).

isometry. A length-preserving transformation.

join of two points. The line joining the two points.

limiting points of two non-intersecting circles α and β. The two common points of any two circles orthogonal to α and β.

line at infinity. Ideal line whose points are centers of pencils of parallels.

medial triangle of $\triangle ABC$. Triangle formed by joining the midpoints of the sides of $\triangle ABC$.

median of a triangle. A cevian through the midpoint of a side.

mid-circle, or circle of antisimilitude. Circle that serves to interchange two given circles by inversion.

n-gon. A polygon with n vertices and n sides.

Napoleon triangle of $\triangle ABC$.
inner. Triangle whose vertices are the centers of equilateral triangles erected internally on the sides of $\triangle ABC$.
outer. Triangle whose vertices are the centers of equilateral triangles erected externally on the sides of $\triangle ABC$.

orthic triangle ($\triangle DEF$) *of* $\triangle ABC$. Triangle whose vertices are the feet of the altitudes of $\triangle ABC$.

orthocenter (H) *of a triangle.* Point of intersection of altitudes.

orthogonal circles. Two intersecting circles whose tangents at either point of intersection are at right angles.

parabola. Conic with eccentricity $\epsilon = 1$, so that O is on α (see definition of *conic*).

Pascal line of a hexagon whose vertices lie on a circle (or on any other conic). Line containing the three points of intersection of pairs of opposite sides of the hexagon.

Peaucellier's cell. A linkage that traces the inverse of a given locus.

pedal triangle of a point P *with respect to* $\triangle ABC$. The triangle formed by the feet of the perpendiculars drawn from a point P to the sides of $\triangle ABC$ (or their extensions).

pencil of circles $\alpha\beta$. Circles orthogonal to two distinct circles orthogonal to α and β.

pencil of lines. All the lines (in one plane) through a point.

point at infinity, P_∞. The ideal common point of all straight lines, regarded as circles in the inversive plane.

polar circle. Circle that reciprocates the vertices of an obtuse-angled triangle into the respectively opposite sides.

polar of a point P with respect to a circle. Line joining intersections $AB \cdot DE$ and $AE \cdot BD$, where AD and BE are two secants (or chords) through P.

pole of a line p with respect to a circle ω, center O. Inverse of foot of perpendicular from O to p. Alternatively, the point of intersection of the polars of any two points on p.

polygon. A closed, broken line in the plane.

power of a point P with respect to a circle. $d^2 - R^2$, where d is the distance from point P to the center of the circle, and R is the radius.

product (or sum, or resultant) of two transformations. The result of applying the first transformation and then the second.

projective plane. Euclidean plane plus one ideal line (see *line at infinity*).

quadrangle. A polygon with 4 vertices and 4 sides.
convex quadrangle. Both diagonals inside.
re-entrant quadrangle. One diagonal inside, one outside.
crossed quadrangle. Both diagonals outside.

quadrilateral. See *quadrangle.*

radical axis of two non-concentric circles. Locus of points of equal power with respect to the two circles.

radical center of three circles with non-collinear centers. Common intersection of all three radical axes, each radical axis taken for two of the three circles.

range of points. All the points on a line.

reciprocation. Transformation of points into their polars, and lines into their poles.

reflection in a line l. A transformation which takes every point into its mirror image, with l as mirror. (See *image.*)

regular polygon. A polygon having a center at the same distance R from every vertex and at the same distance r from every side.

rotation. A transformation resulting from turning the entire plane about a fixed point in the plane.

self-polar triangle. A triangle whose vertices are the poles of the respectively opposite sides.

separation $AC \,//\, BD$ *(for four distinct coplanar points).* Every circle through A and C meets every circle through B and D.

similarity. A transformation that preserves ratios of distances.

Simson line (or simson) of a point P *on the circumcircle of* $\triangle ABC$. Line into which the pedal triangle of P with respect to $\triangle ABC$ degenerates.

spiral similarity (or dilative rotation). The product (q.v.) of a rotation and a dilatation, or vice versa.

stereographic projection. Projection, from O, of sphere through O onto tangent plane at antipodes of O.

sum. See *product.*

topology. Geometry of the group of one-to-one both-ways continuous transformations.

transformation of the plane. A mapping of the plane onto itself such that every point P is mapped into a unique image P' and every point Q' has a unique prototype Q.

translation. A transformation such that the directed segments joining points to their images all have the same length and direction. Alternatively, a dilatation without any fixed point.

tritangent circles of $\triangle ABC$. The four circles tangent to all three sides (or their extensions) of $\triangle ABC$; the incircle and the three excircles.

Varignon parallelogram of a quadrangle. Parallelogram formed by segments joining midpoints of adjacent sides of the quadrangle.

Vector. See *translation.*

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